Department of Education, Ontario

Annual Examinations, 1953

GRADE XIII

PROBLEMS

(To be taken only by candidates writing for certain University Scholarships involving Mathematics)

Ten questions constitute a full paper.

1. Find the locus of the foot of the perpendicular dropped from the focus $(c, 0)$ to a variable tangent of the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$.

2. Given the family of curves

$$9x^2 + 16y^2 - 144 + k(16x^2 - 9y^2 - 144) = 0 \ ,$$

indicate, without finding their coordinates, what points the members of the family have in common and describe how the curve represented by this equation changes as the parameter k varies from large negative to large positive values.

3. Two equal parabolas have the same vertex and their axes are perpendicular. Prove that they have a common tangent and that it touches each parabola at the end of its latus rectum.

4. A parallelogram is formed by drawing tangents at the ends of two conjugate diameters of an ellipse. Prove that the area of the parallelogram is the same for all pairs of conjugate diameters.

5. Given that I_1, I_2, I_3 are the centres of the three escribed circles of a triangle ABC, show that the area of the triangle $I_1I_2I_3$ is $abc/2r$ where r is the radius of the inscribed circle.

6. From two points A and B on a road running north and south, the angle of elevation of a balloon to the east of the road are $45°$ and $30°$, respectively. From a point C half way between A and B, the angle of elevation is $60°$. Give that A and B are 1000 yards apart, find the height of the balloon.

7. The diagonals of a quadrilateral have lengths a and b, respectively, and are inclined at an angle θ to each other. Show that the greatest rectangle which can be drawn with

its sides passing , one through each of the four vertices of this quadrilateral, has area $\frac{1}{2}ab(1+\sin\theta)$.

8. Two uniform spheres, each of radius r and weight W, are placed inside a thin uniform cylindrical tube of radius a, where a exceeds r but is less than $2r$. The tube, which is open at both ends, stands upright on a horizontal table. Show that the tube will not overturn provided its weight exceeds $2W(a-r)/a$.

9. From the pair of equations

$$x = 1 - v + \frac{v}{u} \ , \qquad y = 1 - u + \frac{u}{v} \ ,$$

deduce the pair of equations

$$u = 1 - y + \frac{y}{x} \ , \qquad v = 1 - x + \frac{x}{y} \ ,$$

and conversely.

10. Show that the sum of n terms of the series

$$aA + (a+d)(A+D) + (a+2d)(A+2D) + \cdots$$

is given by

$$\frac{n}{2}\Big[aM + Am\frac{(n-1)(2n-1)}{3}dD$$

where $m = a + (n-1)d$ and $M = A + (n-1)D$.

11. Solve the system of equations

$$x(y+z) = a \ ,$$
$$y(z+x) = b \ ,$$
$$z(x+y) = c \ ,$$

where it is understood that the greatest of a, b, c is less than the sum of the other two.

12. There are four sets of eight cards each. The fours sets are distinguished from one another by the colours of the cards, which are blue, grey, pink, and yellow, respectively. The cards of each set are numbered from 1 to 8. A selection of 4 cards from the 32 cards is to be made, subject to the following requirements:

(a) the colours of the four cards must be different;

(b) the sum of the numbers of the four cards must be 28.

Find the number of ways in which this selection can be made.