GRADE XIII

PROBLEMS

(To be taken only by candidates writing for certain University Scholarships involving Mathematics)

NOTE 1. *Ten questions consitute a full paper.*

NOTE 2. *A supply of squared paper and a book of mathematical tables may be obtained from the Presiding Officer.*

1. (a) If in a certain equation it is found that when x is replaced by $2 - x$ the equation remains unchanged, what can be concluded about the curve represented by the equation?

   (b) If in the equation of a given curve the letters x and y are interchanged, a new equation representing some new curve is obtained. State the relationship between the new curve and the old.

2. Assume that the tangent at a point P on a parabola is the right bisector of the line joining the focus to the foot of the perpendicular from P on the directrix. Deduce that if a parabola rolls without slipping on another parabola of the same size so that the parabolas touch each other at corresponding points, the locus of the focus of the moving parabola is the directrix of the fixed parabola.

3. If an ellipse rolls without slipping on another ellipse of the same size so that so that the ellipses touch at corresponding points, prove that the locus of the focus of the moving ellipse is a circle with centre at one of the foci of the fixed ellipse.

4. Points $A(1,0)$, $B(2,0)$, $C(4,0)$, and $A'(0,1)$, $B'(0,3/2)$, $C'(0,2)$ are given.

   (a) Prove that the intersections of AB' and $A'B$, AC' and $A'C$, BC' and $B'C$ are collinear.

   (b) What may be said concerning AA', BB', CC'?

   (c) State the relationship of the two triangles $AB'C$ and $A'BC'$ in general terms.

5. A man walking along a straight road sees two objects X and Y in line with him in a direction making an angle θ with the road. After walking a distance a along the road he observes that X and Y subtend the angle θ at his eye, and after walking an additional distance b that these objects subtend the same angle θ. Find the distance between X and Y in terms of a, b, and θ.

6. If
$$\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi \ ,$$
prove that
$$x^2 + y^2 + z^2 + 2xyz = 1 \ .$$

7. Three forces act along the sides of a triangle ABC in the directions indicated by BC, CA, and AB. If their resultant passes through the centres of the inscribed and circumscribed circles of the triangle, prove that the magnitudes of the three forces are proportional, respectively, to
$$(s-a)(b-c), \quad (s-b)(c-a), \quad (s-c)(a-b) \ ,$$
where $2s = a + b + c$.

8. (a) Prove the relation $\log_b x = \log_a x \cdot \log_b a$.

   (b) Given that $\log_b 10 = 2.3026$, calculate $\log_{100} b$, correct to *four* places of decimals.

9. In how many ways can 20 pencils be selected from an unlimited supply of 4 different kinds, if at least one of each kind must be chosen?

10. If the roots of $x^3 + px^2 + qx + r = 0$ are in geometric progression, show that $p^3 r = q^3$.

11. Show that when a polynomial $f(x)$ is divided by $(x-a)(x-b)$ the remainder will not contain a constant term if
$$\frac{a}{b} = \frac{f(a)}{f(b)} \ .$$

12. If $f(x) = \frac{ax+b}{cx+d}$, find the condition that $f[f(x)] = x$ for all values of x.