**The quadratic formula**

While students at school get explosed to very few "theorems", particularly in areas other than geometry, they nevertheless have to learn a few formula, which are essentially statements of results.

Even where these formulae may be justified, either in the text or in class, the focus is so much on its application that any justification of it may be completely forgotten. This is unfortunate, as the student can often benefit significanctly more from keeping in mind the reasoning that led up to the formula than from simple memorization of it.

An example of this is the formula for the solution of a quadratic equation:

**The quadratic formula.** *The solutions of the quadratic equation*

$$ax^2 + bx + c = 0$$

*where* $a \neq 0$, *are given by*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \ . \tag{1}$$

At the most basic level, student may simply use this formula to solve particular quadratic equations. It is even possible for them to apply it blindly, not realizing that they can check their solutions by substituting back into the equation. However, if they do rmake such substitutions, then, on empirical grounds, they will undoubtedly come to trust it and apply it mechanically.

At this stage, many students will already have been solving simple quadratic equations with integer coefficients by factorization, and may perceive that there are two independent methods of solving quadratic equations, one that is not guaranteed of success, and the other, use of the formula, which will work all of the time.

However, checking that the formula "works" in specific cases is not enough to establish it. One way to do so is to substitute the values of $x$ given by the formula and verify that indeed they do satisfy the quadratic equation. This is a legitimate proof, but it leaves quite a bit to be desired pedagogically. On the plus side of the ledger, it emphasizes what the formula actually delivers: values of the variable that satisfy the equation. On the minus side, apart from the messiness of the substitution, there is no indication of the significance of the formula, how such a complicated expression might arise, and how it might fit in with other properties and applications of the quadratic. The formula is a black box.

Simply verifying that the formula works has another defect. We do not know that it yields the *only* solutions of the quadratic equation. There may be other numbers that satisfy it, and perhaps we may come across a situation in which these alternatives are what we want.

If our pedagogical goal is that students should exercise control over the mathematics, rather than be led by the nose, we are forced to engage how its results and formulae occur. This leads us to questions of strategy. In the present case, this might mean that we might frame the question differently. Instead of asking, "What is a formula for the solutions of a quadratic equation?" we ask, "How can we solve a quadratic equation?" The second question induces us to think about process rather than product, and to consider how we might start.

For example, we might ask whether there are quadratic equations that are easy to solve. There are two possible answers that we might give. First, we can solve equations when the quadratic is factorable into linear polynomials. Secondly, we can solve quadratic equations of the form $x^2 = k$, when $k$ is positive; indeed, in this case, $x = \pm\sqrt{k}$. Is there any way we can reduce the problem of solving a general quadratic to either of these cases? We note that in fact these are related; the equation $x^2 = k$ can be converted to $0 = x^2 - k = (x - \sqrt{k})(x + \sqrt{k})$.

Most students will probably not know how to proceed from here on their own, and will have to be taught the technique of competing the square. But such considerations will *inform* the technique when it is presented. What makes it easy to solve $x^2 - k = 0$ is the absence of the linear term, and so we need to perform a gambit in effect to absorb the linear term in the general equation. The key recognition is that $ax^2 + bx$ can be rewritten as $a(x^2 + \frac{b}{a}x)$, and that the quantity in parenthesis comprises the first two terms in the expansion of $(x + \frac{b}{2a})^2$ and differs from this expansion by a constant.

Thus we "complete the square"; add a term on the left side to give us the square of a linear polynomial, and then subtract it again, in effect adding 0 to the left side of the equation:

$$a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} = 0 \ . \tag{2}$$

This may be the first time in the career of a secondary student that she has seen in use this general technique of adding and then subtracting a term in an expression, one that is extremely useful and will be seen frequently if she advances her study of mathematics. Already, we have seen one pedagogical advantage of emphasizing the proof, providing an experience of a process that may go far beyond this immediate situation.

From (2), we can make a simple manipulation to either cast the left side as a difference of squares or write the equation in the form $X^2 = K$, and thus reduce the solving of the general equation to dealing with a specific type. But we can draw more from the situation.

The formula focusses our attention on the sign of $D \equiv b^2 - 4ac$, the quantity under the square root. If $D$ is positive, then we know that we have at least two real roots (counting multiplicity). In fact, when $D$ is positive, we can write the equation as a difference of squares, and show that the left side can always be factored as a product of linear factors. From here, it is easy to argue that the solutions we have found are the *only* solution.

Probably the first introduction to imaginary numbers in the syllabus is when students have to deal with quadratic equations for which $D < 0$, where they learn how mathematicians can create new artifacts to impose a general consistency where formerly there seemed to be disparate cases.

By focussing on the process, rather than the formula, students may be able to take advantage of situations where it is more efficient to follow the process rather than apply the formula. For example, given the task of solving $x^2 - 8x - 48 = 0$, and not recognizing a factorization, the student could just as easily render the equation as $(x-4)^2 - 64 = 0$ as apply the formula.

However, there may be other situation that do not fit into the mould where the technique can be used. For example, suppose the equation to be solved is

$$x^4 + 4 = 0 \ .$$

Then we can complete the square in a different way to get

$$0 = (x^4 + 4x^2 + 4) - 4x^2 = (x^2 + 2)^2 - (2x)^2 - (x^2 - 2x + 2)(x^2 + 2x + 2)$$

and solve the equation. Or we might ask, if we can complete the square, why not complete the cube, and apply an analogous technque to solving

$$ax^3 + bx^2 + cx + d = 0 \ .$$

The left side can be written as

$$a\left(x + \frac{b}{3a}\right)^3 + \left(c - \frac{b^2}{3a}\right)x + \left(d - \frac{b^3}{27a^2}\right) = 0 \ .$$

In this way, we can reduce the problem of solving any cubic to solving cubics of the form $x^3 + px + q = 0$, which is the usual starting point for general methods of solving the cubic. In a similar way, we can arrive at $x^4 + ax^2 + bx + c = 0$ as a "canonical form" for equations of the fourth degree.

If we follow the invitation of the proof to consider equations of the third and fourth degrees, we realize that we have developed means of expressing the roots in terms of the coefficients, using the four arithmetic operations along with the extraction of square and cube roots. It is a natural question to ask whether the solution of higher degree equations are attainable from the arithmetic operations and extraction of roots of any order applied to the coefficients.

Delving into the proof reinforces an important perception that students should have about algebra. In any algebraic quest, we are in the business of reading off information from an expression. Sometimes the information can be easily read off, and sometimes it is buried and needs to be brought to light. The purpose of algebraic manipulation is to cast an expression into a form in which the desired information can be drawn. In the case of a quadratic, we have the standard form in descending powers of the variable, the factored form as a product of linear factors and the completion of the square. The factored form allows us to immediately read off its roots. When we use the completion of the square form, as in the left side of (2), while we need an additional step to solve the equation, we can see right away where the quadratic polynomial assumes its maximum or minimum value and exactly what that value is. In fact, we do also get some information about the roots as well. If both $a$ and $4ac - b^2$ are positive, for example, then we can see that the quadratic is positive for all real values of $x$ and so has no real roots.

Thus we see that consideration of the proof has benefits that go far beyond the mere validation of a formula. In the present case, we gain the perception of reducing the general situation to a canonical type, the understanding of how the character of the roots depends on the coefficients, the certainty that the quadratic equation can have no more that two roots, the knowledge of a technique whose range of applicability is wider than the situation at hand, and a broader knowledge of quadratics that can be knitted into a more comprehensive whole.