THE MATCHING GAME

Notes for a lecture given by Ed Barbeau of the University of Toronto to the Royal Canadian Institute on Sunday, December 5, 1999. Enquires can be directed to him at `barbeau@math.utoronto.ca`.

1. A special property of square numbers. A large number of children are standing in line; they face the audience and hold in sequence (one to a child) the numbers $1, 2, 3, \cdots$. All of these numbers are divisible by 1, of course. All children holding numbers divisible by 2 are asked to turn around, so that the even numbers are now hidden from the audience while the odd numbers are still visible. Then all children holding numbers divisible by 3 are asked to turn around; now the multiples of 6 are all visible to the audience, while the odd multiples of 3 are not. Next, all children holding numbers divisible by 4 are asked to turn around; then all numbers divisible by 5; then 6; and so on.

In the long run, it is found that the only children facing the audience are those holding the numbers $1, 4, 9, 16, \cdots$. Is there any reason for this? What is special about these numbers?

These numbers are perfect squares. A child will turn around every time the number called is a divisor of the number that she holds; the child will turn around an odd number of times if she holds a number with an even number of divisors (including 1) and so will end up facing away from the audience. She will turn an even number of times if she holds a number with an odd number of divisors (including 1). Squares are the only numbers with an odd number of divisors.

To see this, we note that we can pair off the divisors of a number: the mate of a divisor is that divisor for which the two divisors multiply to give the number. For example, the divisors of 12 can be paired off: (1, 12), (2, 6), (3, 4), so 12 has evenly many divisors. However, when a number is a square, its square root cannot be paired off with another distinct divisor. For example, in the case of 9, we can form the pair (1, 9), but 3 is a lone wolf; thus, 9 has an odd number of divisors. The same sort of thing happens with 16, 25, 36, and so on.

2. Water and wine problem. You have two vessels: one holds a litre of water and the other a litre of wine. One cubic centimeter is taken from the wine vessel and dumped into the water vessel; the mixture is vigorously stirred. Then from the vessel holding the water mixed with a trace of wine, one cubic centimeter is drawn and mixed thoroughly with the rest of the wine in the wine vessel. At the end, is there more water in the wine vessel, or more wine in the water vessel?

The answer is that there are equal amounts of each fluid in the other's vessel. To see this, note that the water that has been displaced by the wine is now displacing an equal amount of wine in the wine vessel.

3. The card trick of the ten pairs. You ask a subject to select one of ten pairs of cards (1, 2), (3, 4), (5, 6), (7, 8), (9, 10), (11, 12), (13, 14), (15, 16), (17, 18), (19, 20), but to keep it secret. You now gather the cards up in pairs and deal them into four rows (A, B, C, D) of five cards each as follows:

A	1	2	4	6	8
B	3	9	10	12	14
C	5	11	15	16	18
D	7	13	17	19	20

You ask the subject to select the rows that the two cards of her pair now reside in; you can now identify the two cards of the selected pair.

There are ten different ways in which a pair of rows can be determined: (A, A), (A, B), (A, C), (A, D), (B, B), (B, C), (B, D), (C, C), (C, D), (D, D). You just make sure that each selected pair is dealt into a different pair of rows. I have indicated one way to do it. Another, more obfuscating way, is to code the placement of the pairs by the repeated letters in the words: ATLAS, BIBLE, GOOSE and THIGH. In this, you deal the pairs as follows.

A	1	3	5	2	7
B	9	11	10	6	13
C	15	17	18	8	14
D	4	19	12	16	20

4. Matching birthdays. If you have a group of at least 23 people, then there is a better than even chance that you will find two people in the group whose birthdays are on the same day of the year. Try this out on the next gathering of this size that you are in.

5. The Konigsberg Bridges. A famous mathematical problem was solved in the eighteenth century by Leonard Euler (1707-1783). In that century, there were seven bridges which crossed the river Pregel in the town of Konigsberg (once part of East Prussia, now in Russia); they connected two islands in the middle of the river with each other and with the opposite banks.

The townsfolk wondered whether it was possible to plan a continuous walk in which each bridge could be crossed exactly once. Euler showed that it was impossible.

Consider such a walk. It would begin on one piece of land and end on another. For any other piece of land touched only in the middle of the walk, each entrance of the walker would have to be matched by a departure, and these would have to be by means of distinct bridges. the walker would have to enter it and leave it the same number of times using distinct bridges; thus, except for the land you start and end on, each piece of land must have an even number of bridges connecting it elsewhere. This means that at most *two* pieces of land, A, B, C, D, can have an odd number of bridges leading from it. But, as you can see in the diagram, all pieces have an odd number of bridges leading from it. Hence, the walk described in the problem is impossible.

6. Covering a chessboard with dominos. Suppose we have a chessboard (8×8 squares), and a collection of dominos, each large enough to exactly cover two adjacent square of the board. Then it is easy to see that you can use 32 dominoes to cover the board entirely with no overlapping nor uncovered squares. Now let us cut off diagonally opposite corner squares, leaving 62 squares. Can these now be covered by dominoes?

One might think that 31 dominos would do the job. But this is not so. We imagine that the chessboard is coloured in the usual way, with adjacent black and white squares. Each placement of a domino covers a pair of squares: one black with one white. So the 31 dominos must cover an equal number of squares of each colour. However, the two squares that have been cut off have the same colour, so the 62 that remain have two more squares of one colour than of the other.

7. The farmer watering his cow. A farmer wishes to give his cow a drink. By his farm, there runs a straight stream; he takes his jug, walks to the stream and then goes to the cow. The diagram indicates the configuration. How can he do this to minimize the distance that he has to walk from the farmhouse to the cow?

Here we match the actual position of the cow with its position reflected in the axis of the stream to a point on the other side of the stream the same distance from the stream. For each path from the farmer to stream to the cow, there is a path of equal length from the farmer to stream to the reflected position of the cow on the other side of the stream, and *vice versa*. Thus, the problem is equivalent to that of minimizing the distance from the farm to the reflected position of the cow by a path that goes straight to a point on the stream and then straight to the reflected position. This minimizing path is a straight line. Using this, we can reconstruct the path that minimizes the distance (via the stream) from farmhouse to stream to real cow.

8. Partitions of a number. A partition of a positive integer is a set of positive integers whose sum is the given number. For example, the partitions of the number 6 are given by

$$6 = 5 + 1 = 4 + 2 = 3 + 3 = 4 + 1 + 1 = 3 + 2 + 1 = 2 + 2 + 2 = 3 + 1 + 1 + 1$$

$$= 2 + 2 + 1 + 1 = 2 + 1 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1 + 1\ .$$

Thus there are eleven partitions of 6 (including the set consisting of 6 alone). There are four partitions of 6 that involve exactly one or two numbers. There are, also, four partitions of 6 that involves only the numbers 1 and 2. This is no accident.

For any positive integer, the number of partitions that involve only one or two numbers is always equal to the number of partitions that involve only the numbers 1 and 2. To see this, we use a diagrammatical construction that pairs off the partitions of the two types. For example, let us look at partitions of 10. The partition 10 = 7 + 3 can be illustrated by a diagram:

X	X	X	X	X	X	X
X	X	X				

We can pair this off with the partition 10 = 2 + 2 + 2 + 1 + 1 + 1 + 1 which we can illustrate by reading down the columns (instead of across the rows of the diagram). Every partition of 10 as a sum of two numbers can be associated with a distinct partition of 10 as a sum of 1's and 2's. Can you do this for the other possible partitions?

9. Numbers of digits and number of zeros. The number of digits needed to write down all the numbers from 1 up to a given power of 10 inclusive is equal to the number of zeros used in writing down all the numbers from 1 up to the next higher power of 10. For example, to write the numbers from 1 up to 100 inclusive requires 192 digits, and this is the same as the number of 0's that occur when we write the numbers from 1 up to 1000 inclusive. Consider the following table:

Range	*Number digits*	*Number zeros*
1, 10	11	1
1, 100	192	11
1, 1000	2893	192
1, 10000	38894	2893

Can you find a way of proving this equality by "pairing off" occurrences of digits with occurrences of zeros?

10. Pairs of triangles with the same area. How many pairs of triangles can you find all of whose sides have integer length and both triangles of the pair have the same area?

It is easy to find examples by the following artifice. Take a right-angled triangle with integer sides; since the length of one of the arms is always even (why?), the area is an integer. There are two ways in which we can fit two copies of the triangle together to form an isosceles triangle. For example, the (3, 4, 5) right triangle can be used to form (5, 5, 6) and (5, 5, 8) isosceles triangles with the same area.

11. Solving an equation for integers. How many triples (x, y, z) of integers can you find for which the sum of their squares is equal to thrice their product:

$$x^2 + y^2 + z^2 = 3xyz \quad ?$$

It turns out that once we find a single solution, we can go on turning up new solutions indefinitely. It all depends on the theory of quadratic equations. Here is how we proceed.

One solution is $(x, y, z) = (1, 1, 1)$. Can we find any other solution with two ones. Suppose that $y = z = 1$. Then x must satisfy the equation:

$$x^2 - 3x + 2 = 0 \ .$$

This is a quadratic equation and the sum of the roots of this equation is 3. We know one root, $x = 1$; therefore the second root must be 2 and we find the new solution $(x, y, z) = (2, 1, 1)$. We can pair this off with another solution for which $x = 2$ and $y = 1$. For this solution, Z must satisfy the quadratic equation

$$z^2 - 6z + 5 = 0 \ .$$

The sum of the roots of this equation is 6 and one of the roots is 1; the other root must therefore be 5, and we find that $(x, y, z) = (2, 1, 5)$ satisfies the equation.

We can continue on in this way, getting other solutions:

$$(x, y, z) = (2, 29, 5), (13, 1, 5), (2, 29, 169), (433, 29, 5), (13, 194, 5), (13, 1, 34) \ .$$

Can you find others?