THE UNIVERSITY OF TORONTO

UNDERGRADUATE MATHEMATICS COMPETITION

Saturday, March 9, 2002

Time: $3\frac{1}{2}$ hours

`No aids or calculators permitted.`

1. Let A, B, C be three pairwise orthogonal faces of a tetrahedran meeting at one of its vertices and having respective areas a, b, c. Let the face D opposite this vertex have area d. Prove that

$$d^2 = a^2 + b^2 + c^2 \ .$$

2. Angus likes to go to the movies. On Monday, standing in line, he noted that the fraction x of the line was in front of him, while $1/n$ of the line was behind him. On Tuesday, the same fraction x of the line was in front of him, while $1/(n+1)$ of the line was behind him. On Wednesday, the same fraction x of the line was in front of him, while $1/(n+2)$ of the line was behind him. Determine a value of n for which this is possible.

3. In how many ways can the rational 2002/2001 be written as the product of two rationals of the form $(n+1)/n$, where n is a positive integer?

4. Consider the parabola of equation $y = x^2$. The normal is constructed at a variable point P and meets the parabola again in Q. Determine the location of P for which the arc length along the parabola between P and Q is minimized.

5. Let n be a positive integer. Suppose that f is a function defined and continuous on $[0, 1]$ that is differentiable on $(0, 1)$ and satisfies $f(0) = 0$ and $f(1) = 1$. Prove that, there exist n [distinct] numbers x_i $(1 \leq i \leq n)$ in $(0, 1)$ for which

$$\sum_{i=1}^{n} \frac{1}{f'(x_i)} = n \ .$$

6. Let $x, y > 0$ be such that $x^3 + y^3 \leq x - y$. Prove that $x^2 + y^2 \leq 1$.

7. Prove that no vector space over $\mathbf{R}$ is a finite union of proper subspaces.

8. (a) Suppose that P is an $n \times n$ nonsingular matrix and that u and v are column vectors with n components. The matrix $v^T P^{-1} u$ is 1×1, and so can be identified with a scalar. Suppose that its value is not equal to -1. Prove that the matrix $P + uv^T$ is nonsingular and that

$$(P + uv^T)^{-1} = P^{-1} - \frac{1}{\alpha} P^{-1} uv^T P^{-1}$$

where v^T denotes the transpose of v and $\alpha = 1 + v^T P^{-1} u$.

(b) Explain the situation when $\alpha = 0$.

9. A sequence whose entries are 0 and 1 has the property that, if each 0 is replaced by 01 and each 1 by 001, then the sequence remains unchanged. Thus, it starts out as $010010101001 \cdots$. What is the 2002th term of the sequence?

END

Solutions

1. *Solution 1.* Let the tetrahedron be bounded by the three coordinate planes in $\mathbf{R}^3$ and the plane with equation $\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 1$, where u, v, w are positive. The vertices of the tetrahedron are $(0,0,0)$, $(u,0,0)$, $(0,v,0)$, $(0,0,w)$. Let d, a, b, c be the areas of the faces opposite these respective vertices. Then the volume V of the tetrahedron is equal to

$$\frac{1}{3}au = \frac{1}{3}bv = \frac{1}{3}cw = \frac{1}{3}dk \ ,$$

where k is the distance from the origin to its opposite face. The foot of the perpendicular from the origin to this face is located at $((um)^{-1}, (vm)^{-1}.(wm)^{-1})$, where $m = u^{-2} + v^{-2} + w^{-2}$, and its distance from the origin is $m^{-1/2}$. Since $a = 3Vu^{-1}$, $b = 3Vv^{-1}$, $c = 3Vw^{-1}$ and $d = 3Vm^{1/2}$, the result follows.

Solution 2. [J. Chui] Let edges of lengths x, y, z be common to the respective pairs of faces of areas (b,c), (c,a), (a,b). Then $2a = yz$, $2b = zx$ and $2c = xy$. The fourth face is bounded by sides of length $u = \sqrt{y^2 + z^2}$, $v = \sqrt{z^2 + x^2}$ and $w = \sqrt{x^2 + y^2}$. By Heron's formula, its area d is given by the relation

$$\begin{aligned}
16d^2 &= (u+v+w)(u+v-w)(u-v+w)(-u+v+w) \\
&= [(u+v)^2 - w^2][(w^2 - (u-v)^2] = [2uv + (u^2+v^2-w^2)][2uv - (u^2+v^2-w^2)] \\
&= 2u^2v^2 + 2v^2w^2 + 2w^2u^2 - u^4 - v^4 - w^4 \\
&= 2(y^2+z^2)(x^2+z^2) + 2(x^2+z^2)(x^2+y^2) + 2(x^2+y^2)(x^2+z^2) \\
&\qquad - (y^2+z^2)^2 - (x^2+z^2)^2 - (x^2+y^2)^2 \\
&= 4x^2y^2 + 4x^2z^2 + 4y^2z^2 = 16a^2 + 16b^2 + 16c^2 \ ,
\end{aligned}$$

whence the result follows.

Solution 3. Use the notation of Solution 2. There is a plane through the edge bounding the faces of areas a and b perpendicular to the edge bounding the faces of areas c and d. Suppose it cuts the latter faces in altitudes of respective lengths u and v. Then $2c = u\sqrt{x^2 + y^2}$, whence $u^2(x^2 + y^2) = x^2y^2$. Hence

$$v^2 = z^2 + u^2 = \frac{x^2y^2 + x^2z^2 + y^2z^2}{x^2 + y^2} = \frac{4(a^2 + b^2 + c^2)}{x^2 + y^2} \ ,$$

so that

$$2d = v\sqrt{x^2 + y^2} \Longrightarrow 4d^2 = 4(a^2 + b^2 + c^2) \ ,$$

as desired.

Solution 4. [R. Ziman] Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be vectors orthogonal to the respective faces of areas a, b, c, d that point inwards from these faces and have respective magnitudes a, b, c, d. If the vertices opposite the respective faces are $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{O}$, then the first three are pairwise orthogonal and 2$\mathbf{c}$ = $\mathbf{x}\times\mathbf{y}$, 2$\mathbf{b}$ = $\mathbf{z}\times$ $\mathbf{x}$, 2$\mathbf{c}$ = $\mathbf{x}\times\mathbf{y}$, and 2$\mathbf{d}$ = $\mathbf{(z - y)}\times\mathbf{(z - x)}$ = $-$ ($\mathbf{z}\times\mathbf{x}$) $-$ ($\mathbf{y}\times\mathbf{z}$) $-$ ($\mathbf{x}\times\mathbf{y}$). Hence $\mathbf{d}$ = $-$ ($\mathbf{a}$ + $\mathbf{b}$ + $\mathbf{c}$), so that

$$d^2 = \mathbf{d} \cdot \mathbf{d} = (\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c}) = a^2 + b^2 + c^2 \ .$$

2. *Answer.* When $x = 5/6$, he could have 1/7 of a line of 42 behind him, 1/8 of a line of 24 behind him and 1/9 of a line of 18 behind him. When $x = 11/12$, he could have 1/14 of a line of 84 behind him, 1/15 of a line of 60 behind him and 1/16 of a line of 48 behind him. When $x = 13/15$, he could have 1/8 of a line of 120 behind him, 1/9 of a line of 45 behind him and 1/10 of a line of 30 behind him.

Solution. The strategy in this solution is to try to narrow down the search by considering a special case. Suppose that $x = (u-1)/u$ for some positive integer exceeding 1. Let $1/(u+p)$ be the fraction of the line behind Angus. Then Angus himself represents this fraction of the line:

$$1 - \left(\frac{u-1}{u} + \frac{1}{u+p}\right) = \frac{p}{u(u+p)} \ ,$$

so that there would be $u(u+p)/p$ people in line. To make this an integer, we can arrange that u is a multiple of p. For $n = u + 1$, we want to get an integer for $p = 1, 2, 3$, and so we may take u to be any multiple of 6. Thus, we can arrange that x is any of 5/6, 11/12, 17/18, 23/24, and so on.

Comment 1. The solution indicates how we can select x for which the amount of the line behind Angus is represented by any number of consecutive integer reciprocals. For example, in the case of $x = 11/12$, he could also have 1/13 of a line of 156 behind him. Another strategy might be to look at $x = (u-2)/u$, *i.e.* successively at $x = 3/5, 5/7, 7/9, \cdots$. In this case, we assume that $1/(u-p)$ is the line is behind him, and need to ensure that $u - 2p$ is a positive divisor of $u(u-p)$ for three consecutive values of p. If u is odd, we can achieve this with u any odd multiple of 15, starting with $p = \frac{1}{2}(u-1)$.

Comment 2. With the same fraction in front on two days, suppose that $1/n$ of a line of u people is behind the man on the first day, and $1/(n+1)$ of a line of v people is behind him on the second day. Then

$$\frac{1}{u} + \frac{1}{n} = \frac{1}{v} + \frac{1}{n+1}$$

so that $uv = n(n+1)(u-v)$. This yields both $(n^2+n-v)u = (n^2+n)v$ and $(n^2+n+u)v = (n^2+n)u$, leading to

$$u - v = \frac{u^2}{n^2+n+u} = \frac{v^2}{n^2+n-v} .$$

Two immediate possibilities are $(n,u,v) = (n, n+1, n)$ and $(n,u,v) = (n, n(n+1), \frac{1}{2}n(n+1))$. To get some more, taking $u - v = k$, we get the quadratic equation

$$u^2 - ku - k(n^2+n) = 0$$

with discriminant

$$\Delta = k^2 + 4(n^2+n)k = [k + 2(n^2+n)]^2 - 4(n^2+n)^2 ,$$

a pythagorean relationship when Δ is square and the equation has integer solutions. Select α, β, γ so that $\gamma\alpha\beta = n^2 + n$ and let $k = \gamma(\alpha^2 + \beta^2 - 2\alpha\beta) = \gamma(\alpha-\beta)^2$; this will make the discriminant Δ equal to a square.

Taking $n = 3$, for example, yields the possibilities $(u,v) = (132, 11)$, (60, 10), (36, 9), (24, 8), (12, 6), (6, 4),(4, 3). In general, we find that $(n,u,v) = (n, \gamma\alpha(\alpha-\beta), \gamma\beta(\alpha-\beta))$ when $n^2 + n = \gamma\alpha\beta$ with $\alpha > \beta$. It turns out that $k = u - v = \gamma(\alpha-\beta)^2$.

3. *Solution 1.* We begin by proving a more general result. Let m be a positive integer, and denote by $d(m)$ and $d(m+1)$, the number of positive divisors of m and $m+1$ respectively. Suppose that

$$\frac{m+1}{m} = \frac{p+1}{p} \cdot \frac{q+1}{q} ,$$

where p and q are positive integers exceeding m. Then $(m+1)pq = m(p+1)(q+1)$, which reduces to $(p-m)(q-m) = m(m+1)$. It follows that $p = m+u$ and $q = m+v$, where $uv = m(m+1)$. Hence, every representation of $(m+1)/m$ corresponds to a factorization of $m(m+1)$.

On the other hand, observe that, if $uv = m(m+1)$, then

$$\begin{aligned}\frac{m+u+1}{m+u} \cdot \frac{m+v+1}{m+v} &= \frac{m^2 + m(u+v+2) + uv + (u+v) + 1}{m^2 + m(u+v) + uv} \\ &= \frac{m^2 + (m+1)(u+v) + m(m+1) + 2m + 1}{m^2 + m(u+v) + m(m+1)} \\ &= \frac{(m+1)^2 + (m+1)(u+v) + m(m+1)}{m^2 + m(u+v) + m(m+1)} \\ &= \frac{(m+1)[(m+1) + (u+v) + m]}{m[m + (u+v) + m + 1]} = \frac{m+1}{m} .\end{aligned}$$

Hence, there is a one-one correspondence between representations and pairs (u,v) of complementary factors of $m(m+1)$. Since m and $m+1$ are coprime, the number of factors of $m(m+1)$ is equal to $d(m)d(m+1)$, and so the number of representations is equal to $\frac{1}{2}d(m)d(m+1)$.

Now consider the case that $m = 2001$. Since $2001 = 3 \times 23 \times 29$, $d(2001) = 8$; since $2002 = 2 \times 7 \times 11 \times 13$, $d(2002) = 16$. Hence, the desired number of representations is 64.

Solution 2. [R. Ziman] Let m be an arbitrary positive integer. Then, since $(m+1)/m$ is in lowest terms, pq must be a multiple of m. Let $m+1 = uv$ for some positive integers u and v and $m = rs$ for some positive integers r and s, where r is the greatest common divisor of m and p; suppose that $p = br$ and $q = as$, with s being the greatest common divisor of m and q. Then, the representation must have the form

$$\frac{m+1}{m} = \frac{au}{br} \cdot \frac{bv}{as} ,$$

where $au = br + 1$ and $bv = as + 1$. Hence

$$bv = \frac{br+1}{u}s + 1 = \frac{brs + s + u}{u} ,$$

so that $b = b(uv - rs) = s + u$ and

$$a = \frac{sr + ur + 1}{u} = \frac{m + 1 - ur}{u} = v + r .$$

Thus, a and b are uniquely determined. Note that we can get a representation for any pair (u,v) of complementary factors or $m+1$ and (r,s) of complementary factors of m, and there are $d(m+1)d(m)$ of selecting these. However, the selections $\{(u,v),(r,s)\}$ and $\{(v,u),(s,r)\}$ yield the same representation, so that number of representations is $\frac{1}{2}d(m+1)d(m)$. The desired answer can now be found.

4. *Solution.* Wolog, we may assume that $u > 0$, as the arc length for u and $-u$ is the same. The tangent to the parabola at (u,u^2) has slope $2u$, and so the normal has slope $-1/2u$. The equation of the normal is

$$y - u^2 = -\frac{1}{2u}(x - u)$$

and this intersects the parabola at the point

$$\left(-u - \frac{1}{2u}, u^2 + 1 + \frac{1}{4u^2} \right) .$$

The arc length is given by

$$f(u) = \int_{-u-(1/2u)}^{u} \sqrt{1+4x^2}dx = F(u) - F(-u-(1/2u)) ,$$

where F is a function for which $F'(x) = (1+4x^2)^{1/2}$. Then

$$\begin{aligned} f'(u) &= F'(u) - F'(-u-(1/2u))\left(-1 + \frac{1}{2u^2} \right) \\ &= (1+4u^2)^{1/2} - (1+4u^2+4+u^{-2})^{1/2}\left(-1 + \frac{1}{2u^2} \right) \\ &= (1+4u^2)^{1/2} - (4u^4+5u^2+1)^{1/2}\left(\frac{-1}{u} + \frac{1}{2u^3} \right) \\ &= (1+4u^2)^{1/2}\left[1 + \frac{(u^2+1)^{1/2}(2u^2-1)}{2u^3} \right] . \end{aligned}$$

$f'(u)$ is negative when u is close to 0, and positive when u is very large. It vanishes if and only if $2u^3 = -(u^2+1)^{1/2}(2u^2-1)$. Thus $4u^6 = (u^2+1)(4u^4-4u^2+1) \Leftrightarrow 0 = -3u^2+1$, and we have that $f'(1/\sqrt{3}) = 0$. Hence $f(u)$ decreases on the interval $(0, 1/\sqrt{3})$ and increases on the interval $(1/\sqrt{3}, \infty)$. Hence, the arc length is minimized when P is the one of the points

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{3}\right) , \quad \left(-\frac{1}{\sqrt{3}}, \frac{1}{3}\right) .$$

5. *Solution.* Since $f(x)$ is continuous on $[0,1]$, it assumes every value between 0 and 1 inclusive. Select points $0 = u_0 < u_1 < u_2 < \cdots < u_{n-1} < u_n = 1$ in $[0,1]$ for which $f(u_i) = i/n$ for $0 \leq i \leq n$. Then, by the Mean Value Theorem, for each $i = 1, 2, \cdots, n$, there exists $x_i \in (u_{i-1}, u_i)$ for which

$$\frac{1}{n(u_i - u_{i-1})} = \frac{f(u_i) - f(u_{i-1})}{u_i - u_{i-1}} = f'(x_i) .$$

Therefore,

$$\sum_{i=1}^{n} \frac{1}{f'(x_i)} = n \sum_{i=1}^{n} (u_i - u_{i-1}) = n .$$

6. *Solution 1.* Let $y = tx$. Since $x > y > 0$, we have that $0 < t < 1$. Then $x^3(1+t^3) \leq x(1-t) \Rightarrow x^2(1+t^3) \leq (1-t)$. Therefore,

$$\begin{aligned} x^2 + y^2 = x^2(1+t^2) &\leq \left(\frac{1-t}{1+t^3}\right)(1+t^2) \\ &= \frac{1-t+t^2-t^3}{1+t^3} = 1 - \frac{t(1-t+2t^2)}{1+t^3} . \end{aligned}$$

Since $1 - t + 2t^2$, having negative discriminant, is always positive, the desired result follows.

Solution 2. [J. Chui] Suppose, if possible, that $x^2 + y^2 = r^2 > 1$. We can write $x = r\sin\theta$ and $y = r\cos\theta$ for $0 \leq \theta \leq \pi/2$. Then

$$\begin{aligned} x^3 + y^3 - (x-y) &= r^3 \sin^3\theta + r^3\cos^3\theta - r\sin\theta + r\cos\theta \\ &> r\sin\theta(\sin^2\theta - 1) + r\cos^3\theta + r\cos\theta \\ &= -r\sin\theta\cos^2\theta + r\cos^3\theta + r\cos\theta \\ &= r\cos^2\theta\left(\cos\theta + \frac{1}{\cos\theta} - \sin\theta\right) \\ &> r\cos^2\theta(2 - \sin\theta) > 0 , \end{aligned}$$

contrary to hypothesis. The result follows by contradiction.

7. *Solution.* Let r be the minimum number of proper subspaces whose union is the whole of V. Suppose that $S_1, S_2, \cdots, S_r$ are subspaces for which $V = \cup_{i=1}^r S_r$. Note that $r \geq 2$. Since r is minimal, no subspace is contained in a union of the others. Select $v \in S_1 \backslash \cup_{i=2}^r S_i$ and $w \in S_2 \backslash S_1$. For each real λ, $\lambda v + w \notin S_1$. By the pigeonhole principle, there is an index j and two distinct reals μ and ν for which $\mu v + w$ and $\nu v + w$ both belong to S_j. Hence, $(\mu - \nu)v \in S_j$, so that $v \in S_j$. But this contradicts the choice of v.

8. *Solution 1.* Observe that uv^T is an $n \times n$ matrix and that for each column vector w, $v^T w$ is a 1×1 matrix or scalar. Thus, $(uv^T)w = u(v^T w) = (v^T w)u$, and so uv^T is a rank 1 matrix, whose range is spanned

by the vector u. It follows that the matrix uv^TP^{-1} is also rank 1 whose range is the span of u. Note that, for every real scalar λ,

$$(uv^TP^{-1})\lambda u = \lambda u(v^TP^{-1}u) = \lambda(v^TP^{-1}u)u$$

since $v^TP^{-1}u$, being a 1×1 matrix is essentially a scalar. Thus, on the span of u, (uv^TP^{-1}) behaves like $(v^TP^{-1}u)I$, a multiple of the identity. It follow that

$$(v^TP^{-1}uI - uv^TP^{-1})(uv^TP^{-1}) = O\ .$$

Now we are ready to establish the result.

$$\begin{aligned}
&(P + uv^T)(P^{-1} - \frac{1}{\alpha}P^{-1}uv^TP^{-1}) \\
&\qquad = I + uv^TP^{-1} - \frac{1}{\alpha}uv^TP^{-1} - \frac{1}{\alpha}uv^TP^{-1}uv^TP^{-1} \\
&\qquad = I + \frac{1}{\alpha}[(\alpha-1)uv^TP^{-1} - uv^TP^{-1}uv^TP^{-1}] \\
&\qquad = I + \frac{1}{\alpha}[(v^TP^{-1}u)uv^TP^{-1} - uv^TP^{-1}uv^TP^{-1}] \\
&\qquad = I + \frac{1}{\alpha}[(v^TP^{-1}uI - uv^TP^{-1})(uv^TP^{-1})] = I\ ,
\end{aligned}$$

as desired.

Suppose that $v^TP^{-1}u = -1$. Then $v^TP^{-1}uv^T = -v^T = -v^TP^{-1}P$, so that $v^TP^{-1}(uv^T + P) = O$. Since $v^TP^{-1} \neq O$, we cannot have an inverse for $uv^T + P$.

Comment. This is known as the *Sherman-Morrison Formula* (see G.H.Golub, C.F.Van Loon, *Matrix computation*, 1989).

Solution 2. This is similar to Solution 1, with the inverse checked on the right instead of the left. Let $z = P^{-1}u$. Since uv^T is a rank 1 matrix whose range is spanned by u (see Solution 1), $P^{-1}uv^T$ is a rank 1 matrix whose range is spanned by z. We have that

$$\begin{aligned}
(P^{-1}uv^T)\lambda z &= \lambda(P^{-1}uv^TP^{-1})u = \lambda(P^{-1}u)(v^TP^{-1}u) \\
&= \lambda(v^TP^{-1}u)(P^{-1}u) = \lambda(v^TP^{-1}u)z
\end{aligned}$$

so that, on the span of z, $P^{-1}uv^T$ behaves like $(v^TP^{-1}u)I$. Thus

$$(v^TP^{-1}uI - P^{-1}uv^T)(P^{-1}uv^T) = 0\ .$$

Note that

$$\begin{aligned}
&(P^{-1} - \frac{1}{\alpha}P^{-1}uv^TP^{-1})(P + uv^T) \\
&\qquad = I + P^{-1}uv^T - \frac{1}{\alpha}P^{-1}uv^T - \frac{1}{\alpha}P^{-1}uv^TP^{-1}uv^T \\
&\qquad = I + \frac{1}{\alpha}[(\alpha-1)P^{-1}uv^T - P^{-1}uv^TP^{-1}uv^T] \\
&\qquad = I + \frac{1}{\alpha}[(v^TP^{-1}u)P^{-1}uv^T - P^{-1}uv^TP^{-1}uv^T] \\
&\qquad = I + \frac{1}{\alpha}[(v^TP^{-1}uI - P^{-1}uv^T)P^{-1}uv^T]
\end{aligned}$$

from which the result follows.

9. *Solution.* Let us define finite sequences as follows. Suppose that $S_1 = 0$. Then, for each $k \geq 2$, S_k is obtained by replacing each 0 in S_{k-1} by 01 and each 1 in S_{k-1} by 001. Thus,

$$S_1 = 0; \quad S_2 = 01; \quad S_3 = 01001; \quad S_4 = 010010101001; \quad S_5 = 01001010100101001010010101001; \cdots$$

Each S_{k-1} is a prefix of S_k; in fact, it can be shown that, for each $k \geq 3$,

$$S_k = S_{k-1} * S_{k-2} * S_{k-1} \ ,$$

where $*$ indicates juxtaposition. The respective number of symbols in S_k for $k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ is equal to 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378.

The 2002th entry in the given infinite sequence is equal to the 2002th entry in S_{10}, which is equal to the $(2002 - 985 - 408)$th $= (609)$th entry in S_9. This in turn is equal to the $(609 - 408 - 169)$th $= (32)$th entry in S_8, which is equal to the (32)th entry of S_6, or the third entry of S_3. Hence, the desired entry is 0.