INTUITION AND RIGOUR

Notes for a talk given in Vancouver, BC on Friday, April 26, 2002.

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An unfortunate attribute of much commentary on education in mathematics is that themes are seen to be in opposition rather than complementary. Teaching and learning are both complex activities. If both are to be effective, then it is necessary to make sure that they are founded both in clear structures and procedures, as well as in appropriate mental pictures.

Intuition and **rigour** are two characteristics of mathematics education that might appear to be at odds. In reality, they are interrelated. Intuition cannot function reliably unless a suitably rigorous foundation has been laid, nor can rigour be intelligible without the guidance of a sure sense of what properties of the system are significant.

The dictionary defines "intuition" as the art and faculty of knowing without rational proceses, as immediate cognition or as a strange insight. One is said to be following one's intuition when one is unable to offer an explanation, even though the case seems to be compelling. Rigour, on the other hand, is the setting forth of definitions, assumptions and reasoning in a clear, comprehensive and logically sound fashion. In a mathematical endeavour, students must acquire the facility of accessing both intuition and rigour, and the role of the teacher is to provide the tools to enable to the student to do this, taking due account of the experience and the maturity of the student. Intuition that arises without some sense of the mathematical context is just facile reasoning and untutored guesswork; rigour that does not respond to and illuminate current interests of the pupils is just dry pedantry and useless pettifoggery.

The soundness of intuition depends on the worldview and habits of mind that the pupil brings to the situation. Thus, a sound intuition is dependent on the teacher in providing a clear and consistent take on the concepts and processes of mathematics, on the representations placed in front of the pupil, and on the care with which questions and explanations are framed. Through her example of asking questions and paying attention to details, her pupils will slowly be inculcated to develop that curiosity and analytical ability that leads to success (as well as enjoyment and fulfilment) in mathematics. An important ingredient in the growth of intuition is the ability of pupils to articulate what is on their minds, and their willingness to bring hidden processes of thought to light. For the teacher, this implies that there must be a value and emphasis put on communication.

Since I find it more congenial to allow my ideas to evolve out of examples rather than retail generalities, the remainder of my talk will be a discussion with you about some problems and a case study – the teaching of vulgar fractions, an important and tough part of the elementary curriculum.

EXAMPLES

1. **Can the product of two consecutive whole numbers (positive integers) ever be a square?**

Think about this for a minute. Does the question resonate at all with you? Do you even have an opinion on the matter? Would you be willing to place a bet on the answer? Now let us look at such products: $2 = 1 \times 2$, $6 = 2 \times 3$, $12 = 3 \times 4$, $20 = 4 \times 5$. Are any of these squares? Are you developing an intuition about the situation? If you are encountering this question for the first time, you might wish to think further on it before going to the next paragraph.

One way to get an intuition about what is going on is to see how the numbers related to the squares: $1, 2^2 = 4, 3^2 = 9$ and so on. Interpolating the two sets of numbers, we get

$$1, 2, 4, 6, 9, 12, 16, 20, 25, \cdots .$$

Each product of two consecutive whole numbers comes about halfway between two consecutive squares, and this seems to happen long enough for us to believe that this might occur generally. Can this view of the situation be elicited explicitly from pupils? Let us take two consecutive numbers, say 135 and 136. How can we see that 135×136 is not a square: well, it is bigger than $135^2 = 135 \times 135$ and less than $136^2 = 136 \times 136$, and these squares being consecutive have no squares in between.

At this stage, our thinking becomes more rigorous, and we bring to view some aspects of mathematical structure that are relevant; that $1, 2^2, 3^2, 4^2, \cdots$ is a listing of all squares in increasing order, that since $135 < 136$, it follows for example that any multiple of 135 is less than the corresponding multiple of 136. Intuition and rigour become inextricably linked, and it is a tactical decision of the teacher how much intuition ought to be taken for granted and how much needs to be unpacked and analyzed more cogently.

But there is quite a distinct way of looking at the situation, that older children may link into. Two consecutive integers have no common divisor. If the product of two numbers with no common divisor is a square, then it must be that each of the two numbers is itself a square. So to ask whether the product of two consecutive whole numbers is a square is synonymous with asking whether each of two consecutive whole numbers can be a square. Again, we go back to our square sequence $1, 4, 9, 16, 25, 36, \cdots$ and note that the squares become farther apart as we go along, and so we can intuit that we cannot have two squares only one apart. Any of the facts that I have just put on the table may be something that some children will understand right away and others will not see at all; it all depends on their worldview of how numbers behave among themselves.

For teachers of elementary algebra, there is a clear and completely rigorous way to show that two positive squares cannot differ by 1. While I would likely not initiate this with a class, I was once forced into this with a grade 9 class when one student expressed doubt that the statement about squares was true. I put the question to the class, and after they floundered around after a clear explanation, they were ready for the following argument.

Suppose that $1 = a^2 - b^2$, where $a > b \geq 0$, for some integers a and b. The right side of the equation can be factored, so $1 = (a-b)(a+b)$, a representation of 1 as the product of two positive integers. The only possibility is that the two factors are each equal to 1: $a - b = 1$, $a + b = 1$. Solving this gives $a = 1$ and $b = 0$, so that it cannot be arranged that both integers a and b are postive.

This illustrates how the opportunistic teacher can slip in a little solid rigour, when the need for it becomes apparent. It also informs the basic technique of factoring difference of squares, which students may regard otherwise as a little pointless and banal. Here, they have the opportunity to see its power in deftly answering a question.

2. **An ordinary $52-$card deck of playing cards is split into two piles, with all 26 red cards in one pile and all 26 black cards in the second pile. Seven red cards are removed from their pile and shuffled into the black pile. Then seven cards are removed at random from the mixed black-red pile, and put into the pile that was all-red at the beginning. After this is done, are there more black cards in the red pile than there are red cards in the black pile? Or is it the other way around? Or are there exactly as many black cards in the red pile as there are red cards in the black pile?**

What do you think? Do you have any opinion whatsoever? If you do, which of the three alternatives is correct? How you answer this question depends on your worldview of the situation. If you focus on the cards that are transferred back and forth, with all red going across one way and a mixed batch coming back, then there seems to be an assymetry that dictates an affirmative answer to one of the first two questions. But if you focus on the initial and final states, recognizing that at the end, each pile has the same number (26) of cards, then you might ask the question: we may have contaminated the red pile with black cards; where did they come from and what has replaced them, since both piles end up with the same number of cards they started with?

Some of you may be familiar with this problem formulated in terms of vessels filled with equal volumes of water and wine.

3. **In a quiz show, the contestant is offered a prize by being shown three doors, behind two of which is a goat and behind one of which is a car. The contestant, not knowing the situation, will select one of the doors and get the object behind it. The contestant chooses, say, Door A. Before revealing what is behind the door, the quizmaster says, "I am going to give you another chance by opening Door B and showing you that there is a goat behind this door. Now you can either stick with your choice of Door A, or you can switch to the third door, Door C." What should the contestant do? Is there any advantage to switching or to sticking? Or does it matter?**

A few years ago this was a notorious problem that attracted a great deal of heated comment. What is your answer to the question? Again, this depends on how you view the situation. The first thing we should do is to clarify a few things. The contestant really wants the car. The quizmaster always opens a door to reveal a goat, so that the goat does not appear by accident.

What do you think the answer is? The intuition of a lot of people is that, now the contestant has two choices, Door A or Door C, with no information about either of them, so there is an even chance the car is behind either one of them. So it does not make any difference whether you switch of stick. Another view turns on what options are open to the quizmaster. If Door A does have the car, then the quizmaster can pick either of the remaining doors to open. But if Door A has a goat, then the choice of the quizmaster is determined: select the one remaining door that does not have the car. Does this knowledge that the quizmaster might be constrained help the contestant?

To clarify the intuition, let us look at an analogous situation which is more extreme. Suppose that there are ten doors. The contestant selects Door A. The quizmaster then opens Doors B, C, D, E, F, G, H and I and reveals a goat behind each. He then invites the contestant to either stick with Door A or switch to the remaining Door K. Now what should the contestant do?

The point to be made here is that intuitive thinking is not idle thinking. It involves an honest appraisal of your level of understanding, and then seeking out a representation or analogy that might teach your understanding.

4. **Four men, P, Q, R and S, are out walking late at night, having only a flashlight with a weak battery to show the way. They come to a bridge capable of carrying at most two people. Because of the darkness, they can move only with the flashlight close at hand. So to cross the bridge, one or two most go across with the flashlight and then a person must come back with the flashlight, continuing until all are safely across. The four have different maximum speeds. Plodding P requires 10 minutes to get across the bridge; Q requires 5 minutes; R needs 2 minutes, while sprightly S can cross the bridge in 1 minute. When two walk together, they must stay together at the pace of the slower. Given that the four start at one side of the bridge, what is the minimum amount of time required to get all four to the other side?**

Think about this problem for a minute. Most people's intuition leads them immediately to the proposition that the flashlight must be brought back by the fastest of the four, S. This take on the situation is remarkably robust, and there are many who would be prepared to place a substantial bet on 19 minutes being the shortest possible time.

However, the traverse can be accomplished in a shorter time. Intuition needs to be augmented by imagination: what are the possible programs for achieving the task? are we unnecessarily restricting the options? What makes the time large? Whenever P crosses, that uses up 10 minutes, so we better make sure P crosses only once. When Q crosses, that needs 5 minutes, and we better not bring him back with the flashlight, because that will cost 15 minutes altogether. But wait a minute. If we can make Q cross with P, then we need not count Q's time in the total because it is aborbed in Q's 10 minutes. So can P and Q cross together? Now we need to hone or thinking (or increase the rigour of our analysis). How many bridge crossings will there be? Three. So can P and Q go together on the first crossing? Why, or why not? How about the last crossing? What is the remaining possibility? Is this viable?

Once we accept the idea that our intuition needs to be tutored in some way, then we are forced to sharpen our reasoning, to increase the rigour of our analysis.

5. **A square is subdivided into nine pieces by four line segments joining a vertex to a midpoint of a side, as in the diagram below.**

   Suppose that the square has area 1. What is the area of the piece in the middle that does not abut any of the sides?

This is an interesting problem where intuition can possibly lead one astray. Some may feel that the answer is 1/4. How does your intuition respond to that? Once, when I did this problem with a group of students, one student asked whether the middle piece was actually a square. What is your belief? Is it a square? Does this seem obvious? I put the question to the class, who had a great deal of difficulty trying to answer it. After some time, I asked the students whether there was anyone who really had any doubts as to whether the middle piece was a square. Virtually none of them did. Was their intuition faulty, or was there something behind it? Do you have any doubts about it being a square? What is there about the situation that might lead to some assurance that it is a square? Can we uncover what your intuition may be implicitly relying on?

This is a good question to ask, as it leads us to develop the tools that will have us describe our intuition and thus help us to make it probe more deeply and assuredly. The students were trying to draw all sorts of construction lines and proving theorems about triangles. But the key to the situation is *symmetry*; we expect it to be a square because of the symmetry of the situation. How can we describe that symmetry?

The figure is symmetrical in that it is carried to itself by a 90 degree rotation about the centre of the square. This rotation takes each of the internal lines joining a vertex to a side midpoint to an adjacent one; this confirms that the internal lines must meet at right angles, so the inner figure is at least a rectangle. Each vertex of the inner figure is an intersection of two of the inner lines; the rotation takes each such intersection point to an adjacent point of the inner figure. Since a rotation preserves distance, the sides of the inner figure must be equal. Thus, the inner figure has four equal sides and four right angles, and this makes it a square.

The rigour of this explanation in terms of rotations provides both clarity and solidity to what otherwise might be an inchoate sense of what the situation is. But this language of rotations is not something that pupils will find easy to understand or produce, and would have to be preceded by a lot of advanced work in the curriculum. The teachers would have to pave the way long in advance by encouraging pupils to look at geometrical objects in terms of the symmetry that they evince; this might be done for example in an art class in which pupils are encouraged to draw symmetrical figures, describe that symmetry, and to analyze patterns that might appear on clothing or wall decorations.

Having disposed of the issue that the inner figure is square, let us go back to the original question.

There is an insightful way to look at the situation. The remaining eight figures have two distinct shapes; do these shapes bear any relationship? How do you know?

6. **A wall meets the ground at right angles. A ladder is vertically pressed against the wall, with its base where the wall and ground meet. The base then slips along the ground away from the wall, while the top of the latter slides down the wall, until the ladder is flat on the ground. Describe the path of a point exactly halfway down the ladder.**

What does your intuition tell you about this situation? Will the point move in a straight line? Why or why not? Will the point scoop down and follow a bowl-shaped curve? Or will it scoop up and follow a hump-shaped curve? Will the path display any kind of symmetry?

Let us deal with the question of symmetry. If we run the scenario backwards, and have the ladder slide up the wall, then you realize that it is as though we exchanged the roles of the wall and the ground. The symmetry that reverses the wall and ground is a reflection with an axis that makes an angle of 45 degrees with each, so we would expect the path to be symmetrical about this line.

This seems to be a problem where one's intuition can be misled. One can imagine the ladder sliding, and the positions of the latter having an envelope shaped as in the diagram below:

Thus we can surmise that the midpoint of the latter has a path shaped like such an envelope. But let us stop and think for a minute. As the ladder begins its slide away from the way, the base of the ladder is moving in a direction perpendicular to the wall, and we can plausibly see that every higher point on the ladder initial moves in the same perpendicular direction. This seems to militate against the idea that the midpoint scoops down. Similarly, when the whole ladder is about to rest on the ground, it is essentially moving verticall downwards. So it may well be that the path starts off being perpendicular to the wall and ends up being perpendicular to the ground.

To some people, this may seem to be pretty counter-intuitive, so let us see how we make the situation a little more persuasive. Look at the right triangle formed at a given time by the wall, the ground and the ladder. This right triangle constitutes half of a rectangle with the ladder as a diagonal, and the midpoint of the ladder as the centre of that diagonal. Now what do we know about a rectangle? The two diagonals are of equal length and intersect in the midpoint of both. So the line that joins the intersection of wall and ground to the midpoint of the ladder is half of a diagonal of the rectangle, and its length is equal to half the length of the diagonal. This means that at any time, the midpoint of the ladder is distant from the wall-ground corner by a constant amount equal to half the length of the ladder. What is then the path of such a point? Why, it is a quarter circle whose centre is at the wall-ground corner.

It is interesting that, were the ladder to tip over in such a way that the bottom of the ladder remains fixed at the place where the wall and ground meet, the path traced out by the midpoint would be the same quarter circle. The positions of the ladder are the other diagonals of the rectangles mentioned above. This situation is modelled by the legs of some collapsible ironing boards.

A CASE STUDY: FRACTIONS

Fractions is a very big area, and all I can do in the limited space available is to sketch out a few of the issues and suggest alternative approaches.

If one were to identify the a key area in the elementary curriculum, that would have to be ratio and proportion, that branch of mathematics that deals with rates, percentages, units of measurement and scale. The mathematical demands on ordinary citizens generally come in this area, and a firm understanding is required of any pupil who plans to take mathematics at a higher level. The ability to handle ratio problems depends very much on numeracy, a good feel for the number system and its properties. One area that causes particular difficulty for pupils is fractions. This is a topic that can be adequately negotiated only if your students have a firm sense of what fractions are and they attain some kind of facility in manipulating them.

The good news is that the teaching of fractions can be spread over the six years of the primary curriculum, so that pupils have the opportunity to establish a good base of experience with them before you have to systematize the methods of dealing with them. The bad news is that fractions appear in several different contexts, and if one is not careful, pupils can come away with a lot of conflicting and ill-formed ideas as to what they are and how they can be handled. I would like to look at a few aspects of the situation and argue that both intuition and rigour figure prominently and intertwinedly.

What is a fraction? What is the picture that you bring to mind? A pizza cut into pieces? A picture of several balls with some of the balls circled? Marks on a measuring cup or on a ruler? Or do you think of a ratio, a percentage? In introducing children to fractions, you want to create an image that does not confuse separate ideas, that gives a strong sense of the reality of the concept, and that it robust enough to sustain the arithmetic operations that you will need to perform.

The pizza picture seems to be popular, perhaps because of the tenacious belief in educational circles that anything involving food is a surefire draw for the young. But this is actually a needlessly complex situation, at least to begin with, in that it may confuse different contexts. If you cut a pizza into six pieces, what is half the pizza? Three pieces? Or pieces that make up half the area? These need not be the same. Of course, you can get around this by deciding that the pieces will be of equal size, but then do you want to restrict your divisions of pizza in this way and will you be paying a price of a lack of flexibility later on? Are you willing to cope with area and fraction at the same time? If students have a conceptual difficulty with one, will it sabotage their ability to grasp the other?

An important issue in dealing with fractions is what the unit is. The glossing over of this point is probably responsible for some difficulty and confusion that pupils have with fractions. So you need a picture that introduces the unit in a natural way. You want a model that identifies fractions as numbers, and one in which admits a natural way of interpreting addition, subtraction, multiplication and division.

The number line can be introduced naturally to young children through the medium of a tape measure. There are probably many households in which five- or six-year-olds keeps track of their growth by a pencil mark on the frame to the kitchen door. The measurement of a continuous quantity such as height immediately raises the issue of fractions of a unit, and the comparison of growth leads to a natural way of looking at addition and substraction in terms of concatenation, the laying of lengths side by side.

In this way, we can build intuitions about addition of fractions in particular and numbers in general - that the sum of two positive numbers should be bigger than either of them. One this view is firmly established, and we have now begun to address the practicalities of adding fractions, we can test our rules and algorithms against our intuition. For example, if we want to add $\frac{2}{5}$ and $\frac{3}{8}$, is $\frac{(2+3)}{(5+8)} = \frac{5}{13}$ a suitable result? If we can find the summands on the number line, where should the sum be? But where would the fraction obtained by adding the numerators and the denominators be?

There are two alternative models for multiplication. One is to think of what multiplication does to the number line. For example, let us look at multiplication by 3. Multiplying by 3 takes 2 to 6, 7 to 21, 1 to 3 and 0 to 0. Geometrically, it amounts to a stretch (the technical word is *dilation*) of the line by a factor of 3, keeping 0 fixed. In otherwords, we expand the line to scale. 3×2 is that number that bears the same ratio to 2 as 3 bears to 1.

Within this framework, we can give sense to multiplication by $\frac{2}{3}$, say. Again it is represented by a dilation of the line. Multiplying 1 by $\frac{2}{3}$ should give the result $\frac{2}{3}$ (this is what multiplying with 1 as a factor does), so the line shrinks to scale with 0 going to 0 and 1 going to $\frac{2}{3}$. Where should this shrinkage to scale take 6, say? or 12? or $\frac{7}{9}$? If we take the time to establish this intuitive geometric view of what multiplication *signifies*, then it becomes natural to develop the rules that will deliver it.

A second view of multiplication can be given in a more combinatorial way, starting with the idea of multiplication as successive addition of the same number. For example, 5×8 can be thought of as $8 + 8 + 8 + 8 + 8$ and diagrammed by a rectangular array of dots with eight dots in each of five rows:

To extend this to fractions, we can move into realm of area. A 5×8 rectangle can be subdivided into 5 rows of 8 unit squares, and its area seen to be $5 \times 8 = 40$ square units. So we can model multiplication in terms of the calculation of areas of rectangles; to get a handle on what 5×8 is, think of a rectangle with sides 5 and 8 and imagine its area.

We have already seen that fractional lengths make sense, so we can now envisage a multiplication of fractions in terms of a rectangle. For example, consider $\frac{2}{3} \times \frac{4}{7}$. Take a unit square, and, by subdividing its sides, cut out from it a rectangle of sides $\frac{2}{3}$ and $\frac{4}{7}$. We can subdivide it into $21 = 3 \times 7$ small rectangles, each of area $\frac{1}{21}$; our rectangle requires $8 = 2 \times 4$ of them, so the area of the rectangle (and the product of the fractions) is $8 \times \frac{1}{21} = \frac{2 \times 4}{3 \times 7} = \frac{8}{21}$.

We can go on to provide in this way a worldview of the whole panopy of fractions. There are many things I have not talked about: negative as well as positive numbers, division, the relationship between ratios and fractions. Suffice it to say that a proper reference to the artifacts of everyday life can provide the mental representations of fractions upon which a solid intuition can be founded, and lead naturally in due course to the formulation (ideally, by the pupils themselves) of the rules that govern their manipulation.

Intuition and rigour in mathematics are closely related. Fluency requires the ability to conjure up the right sort of pictures, the fruitful way of thinking about something, along with the ability to take things apart, to think formally and logically, at those points when the intuition is uncertain. The challenge for the teacher is ensure that a clear focus is kept in this mathematical picturing and that pupils are always encouraged to keep a sense of the reality of the mathematical artifacts, even when they are behaving in the most mechanical way. The communication fostered in the posting and answer of questions is the bond that keeps the two in touch.