# POLYNOMIALS WITH NO POSITIVE ROOTS.

*A mathematical vignette*

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In this investigation, we will consider polynomials with real coefficients. To avoid trivial complications, we will suppose that the leading coefficient is 1 and that the constant term is nonzero, so that the polynomial $p(x)$ has the form

$$p(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0,$$

with $a_0 \neq 0$. If we substitute a positive real value for $x$, then the value of the polynomial is greater than 0. Thus, the polynomial has no positive roots.

However, for polynomials of degree greater than 1, it is possible for the polynomial to have no positive roots while some of its coefficients are negative. An example is $x^2 - x + 1$.

This raises the question: *given a polynomial $p(x)$ as described that has no positive roots, it is possible to find a polynomial $q(x)$ for which all the coefficients of $p(x)q(x)$ are nonnegative?*

## 1. An example.

Let

$$p(x) = x^6 - x^4 + 2x^3 - x^2 + 1.$$

Begin by showing that this polynomial has no positive roots. There are many ways of doing this by rearranging the terms and finding combinations to show that this always assumes a positive value when $x$ is positive.

We can try a multiplier $q(x) = x + u$, where $u$ is a positive real. Show that $p(x)q(x)$ always has a negative coefficient. The next step is to try $q(x) = x^2+bx+c$. In this case, we can make all the coefficients of the product nonnegative. Find a cubic polynomial $q(x)$ for which all the coefficients of $p(x)q(x)$ are positive.

Begin by showing that this has no positive roots. The multiplier $q(x)$ cannot be linear, but there are quadratic multiplers $q(x)$ for which all the coefficientsof $p(x)q(x)$ are nonnegative. If we want all the coefficients of the product to be positive, then we need a cubic $q(x)$.

Here are some possible paths to solutions. Observe that

$$p(x) = (1 - x^2)(1 - x^4) + 2x^3 = (x^2 - 1)(x^4 - 1) + 2x^3.$$

Two quadratic multipliers are $x^2 + x + 1$ and $x^2 + 2x + 1$. A cubic multiplier is $x^3 + 2x^2 + 2x + 1$.

## 2. Quadratic polynomials.

The fundamental theorem of algebra says that any monic polynomial $p(x)$ with real coefficients can be written as a product of factors of the form $x+u$ and $x^2-bx+c$, where $u, b, c$ are real numbers and $b^2 < 4c$, *i.e.* the quadratic has no real roots and so is irreducible over the real numbers. Since we are assuming that the constant term of $p(x)$ is nonzero, so also are all the terms $u$ and $c$ in the product.

If $p(x)$ has no positive roots, then $u$ must be positive. So one strategy to find a multiplier $q(x)$ is to find a multiplier that works for each quadratic factor $x^2-bx+c$ and take their product to get a multiplier for $p(x)$.

For an irreducible quadratic, the constant $c$ is always positive, but $b$ can be positive or negative. If $b$ is negative, then all coefficients are positive and the multiplier can simply be the constant 1. So we need to consider only the case $p(x) = x^2 - bx + c$, where $b$ is positive and $b^2 < 4c$.

Under what circumstances is there a linear polynomial $x + u$ such that all the coefficients of $(x + u)(x^2 - bx + c)$ is positive?

We can start the investigation with a specific example $x^2-x+1$. Now generalize to $p(x) = x^2-ax+a$ where $0 < a < 4$. One strategy is to multiply by a high enough power of $(x + 1)$. The motivation behind this is that each such factor introduced a new root $-1$ along with the roots of $p(x)$ whose sum and product are both $a$. The coefficients of $p(x)q(x)$ can be expressed in terms of the symmetric functions of its roots.

Look at different values of $a$ and see how large a value of $k$ is needed for $(x^2 - bx + c)(x + 1)$ to have all positive coefficients.

For the general polynomial $p(x)$, finding a multiplier by taking the product of the multipliers for its quadratic factors seems extravagant and would likely produce a polynomial $q(x)$ whose degree is higher than necessary. This raises the question of determining the minimum degree of $q(x)$ to ensure that all the coefficients of $p(x)q(x)$ are nonnegative/positive.

Here are some specific results:

Let $b > 0$ and $b^2 < 4c$. Then

$$(x^2 - bx + c)(x + u) = x^3 + (u - b)x^2 + (c - bu)x + cu,$$

and we can find a value of $u$ that works provided $b^2 < c$. In this case, take $b < u < c/b$.

Note, for example, that

$$(x^2 - x + 1)(x + 1) = x^3 + 1;$$

$$\left(x^2 - \frac{3}{2}x + \frac{3}{2}\right)(x + 1)^3 = x^5 + \frac{3}{2}x^4 + x^2 + 3x + \frac{3}{2};$$

$$(x^2 - 2x + 2)(x + 1)^4 = x^6 + 2x^5 + 5x^2 + 6x + 2.$$