THE THEATRE LINE PROBLEM.

A mathematical vignette

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Here is a standard type problem from a Greade 9 textbook: *A man is standing in a theatre line. 5/6 of the people in the line are in front of him, and 1/7 of the people in the line are behind him. How many people are there in the line?*

It is clear from the position of the problem in the book that an algebraic solution is intended. For example, student could let x denote the total number of people in the line, then set up and solve the equation

$$\frac{5}{6}x + \frac{1}{7}x + 1 = x.$$

However, not every student may see the problem in this light. There may be one who recognizes at once that there must be 42 people in the line. Why? The answer must be a multiple of both 6 and 7, and in fact the smallest such multiple: 42.

This is an argument that some will see right away, whereas for others, it take a little while for the penny to drop. However, there is more to this approach than meets the eye. Because multiplying the number in the line by 5/6 and 1/7 must produce an integer, it is easy to see that the number must be a multiple of 42. Why the smallest? If we try a particular multiple, the number of people remaining in the line after we remove $5/6 + 1/7$ is propostional to the multiplier, so that it increases with the multiple chosen. Since this number is to be 1, we must look at 42.

But all we know at this point is that the number of people remaining in the line is smallest when there are 42 people altogether, rather than a larger multiple. However, we do not know that this number is 1, which the problem requires. So to simply give 42 as the answer it not enough; we have to check that it actually works. (And it does!)

To see the importance of this checking, suppose we repose the problem, replacing the fractions by 2/3 and 1/5. Then the above argument suggests that the answer for the total number in the line must be 15. But 2/3 of 15 is 10 and 1/5 of 15 is 3, accounting for 13 people, and leaving a remainder of 2.

How is this reflected in the algebraic equation we might consider? The equation

$$\frac{2}{3}x + \frac{1}{5}x + 1 = x$$

is indeed solvable, but the solution $x = 7/2$ is not an integer. Thus, the problem has no solution.

This approach to the problem opens up a line of enquiry. Let us generalize: *A man is standing in a theatre line. The fraction u of the line is in front of him and the fraction v is behind him, where $u + v < 1$. How many people are in the line?* For what values of the pair (u, v) is there an integer solution to the determination of the number of the people in the line?

Since it is difficult to approach the problem on purely algebric terms, it might be best to start with specific examples.

(1) We can make a slight generalization to begin with, and note that the problem has a solution (the product of the denominators) when $(u, v) = (2/3, 1/4)$, $(3/4, 1/5)$, $(4/5, 1/6)$, $\ldots$, $((m-1)/m, 1/(m+1)$.

More generally, if we let

$$(u, v) = \left(\frac{m-r}{m}, \frac{r}{m+1}\right)$$

and try $m(m+1)$ as the total number in the line, then the number remaining in the line after we remove the two fractional parts is r.

On the face of it, the fraction pair does not yield a solvable problem when $r > 1$. But it may be that the fractions for u and v are not in lowest terms. Suppose $r = st$ where m is a multiple of s and $m+1$ of t: $m = si$, $m + 1 = tj$. Then

$$(u, v) = \left(\frac{i-t}{i}, \frac{s}{j}\right)$$

and $ij - [j(i-t) + is] = 1$.

(2) We can test the waters by fixing u to be a specific fraction. Suppose, for example, the $u = 1/2$. Then some experimentation will provide infinitely many possible values of v: $\{1/3, 2/5, 3/7, \ldots\}$ In fact, $v = x/(2x+1)$ will provide a solvable problem for every integer x. Similarly, $(u, v) = (1/3, (2x+1)/(3x+2))$ will allow for a solvable problem. Looking at other examples will reveal othr patterns.

Other directions of investigations can be given by such problems:

(1) Replace the man by a couple or a group of a fixed size.

(2) Given two fractions whose sum is less than one, ask for the size of the smallest group for which the problem has an answer.