Sets of integer additions.

A mathematical vignette

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1. The starting problem.

Here is a problem that can be used to initiate a student investigation:

Replace the letters by the nine digits from 1 to 9 to obtain valid sums in the following equations:

$$A = B + C; \qquad D = E + F; \qquad G = H + I.$$

This may seem a little unfair, because try as the students may, there is no solution to the problem. This can be discovered by not too much trial and error, But we can ask whether there is a quicker way to see that the task is impossible.

There are two possible ways to see this. The first is to note that the sum $A+D+G$ must be equal to the sum of the other six digits, making the total sum of all digits a multiple of 2. But the sum of the digits is 45, making this impossible.

The second way is to observe that in each sum, either none or two of the three digits must be odd, so that the total number of odd digits in the three sums has to be even. But there are five odd digits to be placed: 1, 3, 5, 7, 9.

2. Related problems.

Suppose that we consider a more general question. Given any 9 consecutive integers, is it ever possible to make substitutions for $A, \ldots, I$ to get correct sums. What is the situation for example for the numbers from 2 to 10 inclusive? Can 0 be one of the integers?

Another way to generalize the problem is to take any positive integer k, and partition the numbers from 1 up to $3k$ inclusive into triplets $(a, b; c)$ for which $a + b = c$.

In the sections that follow, we give some partial results. You are encouraged to investigate further before reading on.

3. Nine consecutive integers.

If the nine integers include the number 0, then it must include both the numbers 1 and -1, and 0 must be the sum of a and $-a$ for some integer a.

We can solve the problem for the integers from 2 to 10, inclusive:

$$(2, 6; 8), (3, 7; 10), (4, 5; 9).$$

Suppose that all the nine integers are positive and that the smallest number is a and the largest is $a+8$. The number a must be one of the two summands and the sum of the two numbers that involve it must be at least $a+(a+1) = 2a+1$. This sum must not exceed $a+8$, so that $a \leq 7$. Furthermore, the numbers $a, a+1, \ldots, a+8$ must have an even number of odd numbers, so that a must be even. Therefore $a = 2, 4, 6$. We have already handled the $a = 2$ case.

When the smallest number is -2, we can find the example: $(-2, 2; 0), (-1, 4; 3), (1, 5; 6)$. When $a = -4$, we have $(-4, 4; 0), (-2, -1; -3); (1, 2; 3)$. We can derive a set of triples for $a = -6$ from those for $a = -2$: $(-5, -1; -6), (-4, 1; -3), (-2, 2, 0)$.

4. Sums involving the integers from 1 to $3k$.

As in section 1, for the solution to exist, the number of odd numbers between 1 and $3k$ inclusive must be even. When k is odd, there are $\frac{1}{2}(3k+1)$ odd integers and so there is no solution when $3k+1$ is twice an odd number. When k is even, there are $(3k)/2$ odd integers and there is no solution when $3k$ is twice an odd integer.

This rules out a solution when $k = 2, 3, 6, 7, 10, 11, \ldots$, *i.e.* when $k = 4r+2$ or $k = 4r+3$.

Is there always a solution for other values of k. Here are some examples where $(a, b; c)$ denotes a triple where $a + b = c$:

$k = 1$: (1, 2; 3)

$k = 4$: (1, 11; 12), (2, 6; 8), (3, 7; 10), (4, 5; 9)

$k = 5$: (1, 8; 9), (2, 13; 15), (3, 11; 14), (4, 6; 10), (5, 7; 12)

$k = 8$: (1, 10; 11), (2, 19; 21), (3, 12; 15), (4, 18; 22), (5, 9; 14), (6, 17; 23), (7, 13; 20), (8, 16; 24)