A diophantine equation.

A mathematical vignette

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0. The equation.

A diophantine equation is a polynomial equation in any number of variables for which solutions in integers are sought. Under investigation in this vignette is the equation:

$$x^4 + y^3 + z^2 - 3xyz = 0. \qquad [1]$$

Since this is the equation of a surface in three-dimensional space, there will be infinitely many solutions in real numbers. However, it is not clear whether there are any solutions where x, y, z take integer values, or even rational values.

1. Preliminary spadework.

The first thing we should do is to see whether there are any solutions at all. The search for a solution often involves imposing some restriction that will lead to a simpler equation.

(1) Find solutions where one of the variables takes the value 0.

(2) Are there any solutions for which $x = y = z$?

(3) Are there any "obvious" solutions?

(4) Try to find solutions for which at least one of the variables assumes the value 1; the value 2.

The idea is to find as many numerical solutions as you can and look for patterns that may help you in generating other solutions. One useful observation is that the polynomial is quadratic in the variable z, so that the equation can be rewritten as

$$z^2 - (3xy)z + (x^4 + y^3) = 0. \qquad (1)$$

Suppose that we have found a solution $(x, y, z) = (u, v, w)$ in integers. Then $z = w$ is one solution of the quadratic equation

$$z^2 - (3uv)z + (u^4 + v^3) = 0.$$

This quadratic will have a second solution $z = w'$. Since

$$w + w' = 3uv,$$

it follows that w' is an integer. (Note also that

$$ww' = u^4 + v^3.)$$

(5) For all the solutions you have found so far, find the related solutions with the same values of x and y and a different value of z.

2. The first family of positive solutions.

Two particular solutions of the equation are $(x, y, z) = (1, 1, 2), (2, 4, 4)$. This suggests that we might look for those solutions for which $y = x^2$. Then we have to solve

$$z^2 - 3x^3 z + (x^4 + x^6) = 0. \qquad [2]$$

(6) Consider [2] as an equation in z. Show that its discriminant is equal to $x^4(5x^2 - 4)$ and deduce that there is a solution in integers where x is chosen so that $5x^2 - 4$ is the square s^2 of some integer s.

(7) Find some integers x for which $5x^2 - 4$ is a square, and so determine more soltions for equation [1] in positive integers.

We are led to consider the diophantine equation

$$s^2 - 5x^2 = -4. \qquad [3]$$

This particular type of equation occurs frequently in number theory and there is a very elegant theory behind its solution. To set this up, we need to talk about surds of the form $s + x\sqrt{5}$.

(8) Define the *norm* N of the surd $s + x\sqrt{5}$ to be the product of the surd and its *conjugate* $x - x\sqrt{5}$:

$$N(s + x\sqrt{5}) = (s + x\sqrt{5})(s - x\sqrt{5}) = s^2 - 5x^2.$$

Verify that

$$\begin{aligned} N((s_1 + x_1\sqrt{5})(s_2 + x_2\sqrt{5})) &= N(s_1 s_2 + 5x_1 x_2 + (s_1 x_2 + s_2 x_1)\sqrt{5}) \\ &= N(s_1 + x_1\sqrt{5}) \cdot N(s_2 + x_2\sqrt{5}). \end{aligned}$$

The implication of this is that if we can find one solution of equation [3], which asserts that $N(s + x\sqrt{5}) = -4$, we can obtain other solutions of the same equation by multiplying the surd involved by a surd of norm 1.

(9) Find the solution of $s^2 - 5x^2 = -4$ for which s and x have the smallest positive values. Verify that

$$N(\tfrac{3}{2} + \tfrac{1}{2}\sqrt{5}) = 1.$$

Use these facts to find an infinite sequence (s_n, x_n) of positive integer pairs $(n \geq 1)$ for which $s_n^2 - 4x_n^2 = -4$. This sequence of pairs will be defined by

$$s_{n+1} + x_{n+1}\sqrt{5} = (\tfrac{3}{2} + \tfrac{1}{2}\sqrt{5})(s_n + x_n\sqrt{5}).$$

Determine and check numerical solutions for [1] for small values of n.

(10) Write out the first few terms of the sequence $\{x_n\}$. Make and prove a conjecture about the relationship between each x_n and its two predecessors in the sequence for $n \geq 3$.

3. The second family of positive solutions.

It is not hard to find solutions of [1] for which $x = y$ and it may be worth imposing this as a restriction towards the goal of finding a family of solutions with this property. So we support that $x = y$ and $z = vx$, an integer multiple of x. Now the equation to be solved becomes

$$0 = x^4 + x^3 + v^2x^2 - 3vx^3 = x^2[x^2 - (3v - 1)x + v^2]. \qquad [4]$$

(11) We need to find integer solutions for the quadratic equation in x:

$$x^2 - (3v - 1)x + v^2 = 0.$$

For this to be possible, we need that its discrimant be a perfect square. Prove that this happens if and only if $(5v - 1)(v - 1)$ is a perfect square. Determine small positive integer values of v for which this happens and use these to obtain solutions of [1].

4. Solutions where the variables are not necessarily nonnegative.

Note that (x, y, z) satisfies equation [1] if and only if $(-x, y, -z)$ satisfies the equation, so we might as well assume that $x > 0$.

(12) Investigate solutions of [1] when $(x, y) = (u - u)$ and $(x, y) = (u, -u^2)$ for some positive integer u.

5. Discussion.

In this discussion, we exclude the trivial solution $(x, y, z) = (0, 0, 0)$. An investigation such as this can be carried on at different levels, depending on the interest and capability of the students. It integrates topics from the syllabus, including arithmetic (and use of the pocket calculator), pattern recognition and articulation, algebraic manipulation, quadratic equations and surds. But apart from whatever topics this impinges on, it foster important mathematics values such as being alert to structure and picking out ingredients of a situation that may be significant, and illlustrating the power of mathematics in finding solutions to problems that might seem inaccessible.

Some of the work below leads to conjectures that may be beyond the scope of a typical mathematical class to settle. However, these may be useful for students with a special interest and talent, or could be used for enrichment.

(1) There are no nontrivial solutions when $y = 0$, since then the left side of [1] would contain only positive terms. The condition $z = 0$ leads to $x^4 = -y^3 = (-y)^3$,

so that each side is both a fourth power and a cube. This leads to $(x,y,z) = (u^3, -u^4, 0)$ where u is any positive integer.

(2) The condition $x = y = z$ leads to $x^2(x-1)^2 = 0$ and the solution $(x,y,z) = (1,1,1)$. There is a second solution with the same x-values, obtained by solving the quadratic equation $0 = z^2 - 3x + 2 = (x-1)(x-2)$, namely $(x,y,z) = (1,1,2)$.

(3) (4) (5) Some solutions that could be found by inspection and by solving the quadratic in z when the values of x and y are known are

$$(x,y,z) = (1,2,3), (1,-2,1), (1,-2,-7), (2,4,20).$$

(6) (7) By checking small integers, it can be discovered that $5x^2 - 4$ is a square when $x = 1, 2, 5$. These correspond to the solutions

$$(x,y,z) = (1,1,1), (1,1,2), (2,4,4), (2,4,20), (5,25,50), (5,25,325).$$

In trying to find patterns, we observe that $4 = 4 \times 1$, $20 = 4 \times 5$, $50 = 25 \times 2$ and $325 = 25 \times 13$, where the multipliers of the y-value are $1, 2, 5$. This suggests checking out $5x^2 - 4$ when $x = 13$. Indeed, $5 \times 13^2 - 4 = 841 = 29^2$, and this leads us to the solutions

$$(x,y,z) = (13, 169, 845) = (13, 169, 169 \times 5)$$

and

$$(x,y,z) = (13, 169, 5746) = (13, 169, 169 \times 34).$$

(9)(10) We get the sequence $\{(s_n, x_n) : n \geq 1\}$ of solutions:

$$(1,1), (4,2), (11,5), (29,13), (76,34).$$

By inspection, we can conjecture that $x_{n+1} = 3x_n - x_{n_1}$ and solutions of equation [1] are given by

$$(x,y,z) = (x_n, x_n^2, x_n^2 x_{n-1}), (x_n, x_n^2, x_n^2 x_{n+1}^2).$$

Note in passing that the sequence $\{x_n\}$ picks up alternate terms of the Fibonacci sequence. Note also that $s_{n+1} = 3s_n - s_{n-1}$ seems to be true as well. These conjectures can be left to the more enthusiastic students to establish.

(11) It can be checked that $5v - 1$ and $v - 1$ are both squares when $v = 1$, $v = 2$, and $v = 10$. These lead to the solutions

$$(x,y,z) = (1,1,2), (1,1,4), (4,4,8), (4,4,40), (25,25,250), (25,25,1625)$$

respectively. We note that $8 = 4 \times 2$, $40 = 4 \times 10$, $250 = 25 \times 10$, and $1625 = 25 \times 65$. The appearance of the factors 2 and 10 is suggestive, so we can check out $v = 65$. We find that $5 \times 65 - 1 = 324 = 18^2$ and $65 - 1 = 8^2$.

Let us list the first few values of $(v, v-1, 5v-1)$:

$$(1, 0^2, 2^2), (2, 1^2, 3^2), (10, 3^2, 7^2), (65, 8^2, 18^2).$$

This should allow us to conjecture that there are infinitely many values of v for which $(v-1)(5v-1)$ is square and lead us to an infinite family of solutions to equation [1].

(12) When $(x, y) = (u, -u)$, then equation [1] becomes $z^2 + 3u^2z + (u^4 - u^3) = 0$ and its discriminant is $u^3(5u + 4)$. This is a square if and only if $u = v^2$ and $5u + 4 = w^2$, *i.e.* when $w^2 - 5v^2 = 4$. This can be analyzed similarly to the previous cases. We find that $(w, v) = (3, 1), (7, 3), (18, 8), (47, 21)$. Some solutions obtained this way are $(1, -1, 0)$, $(1, -1, -3)$, $(9, -9, -27)$, $(9, -9, -216)$.

When $(x, y) = (u, -u^2)$, then equation [1] becomes $z^2 + 3u^3z + (u^4 - u^6) = 0$, with discriminant equal to $u^2(13u^2 - 4)$. If $v^2 = 13u^2 - 4$, then we have to analyze the diophantine equaation $v^2 - 13u^2 = -4$. This has infinitely many solutions, the smallest of which are $(v, u) = (1, 1), (36, 10)$.