SQUARES OF THE FORM $ab + k$

A mathematical vignette

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Look at the triple of integers: $(1, 3, 8)$. It has the property that the product of any pair of them plus 1 is a perfect square. Similarly, the the triple of integers, $(1, 2, 5)$, the product of any pair of them minus 1 is a perfect square.

How many triples (a, b, c) of integers are there for which the products $ab+1$, $cb+1$, $ac+1$ are all perfect squares? We can ask the same question with the plus sign replaced by a minus sign.

This is a particular rich area of investigation for students, because, one a few examples have been found, there are a variety of ways in which students can identify patterns as well as obtain and extend general results. It is important in this vignette that the students not be denied the pleasure of discovery. The material that appears below is provided as a resource to the teacher, and should not be presented to the student except to put her discovery into context and provide the tools for further research.

A more general question is, given an integer k, determine triples (a, b, c) for which $ab + k$, $bc + k$, $ca + k$ are all squares. We can also ask whether we can find sets of integers with more than three entries for which the product of any pair plus k is square.

1. Some preliminary investigations. By looking at particular cases, you might note that, if you look at the Fibonacci sequence:

$$\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, \cdots\}$$

you will find that successive triples of alternate terms give a family of examples. Thus, for (a, b, c) with $k = 1$, we have $(1, 3, 8)$, $(3, 8, 21)$, $(8, 21, 55)$ and so on, while with $k = -1$, we get $(1, 2, 5)$, $(2, 5, 13)$, $(5, 13, 34)$, and so on.

With $f_0 = 0$, $f_1 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 1$, the validity of these triples depends on the indentities

$$f_n f_{n+2} + (-1)^n = f_{n+1}^2$$

and

$$f_{n-2} f_{n+2} + (-1)^n = f_n^2.$$

Here are a few families that might be discovered:

(a, b, c)	$ab + 1$	$bc + 1$	$ca + 1$
$(r - 1, r + 1, 4r)$	r^2	$(2r - 1)^2$	$(2r + 1)^2$
$(1, (r - 1)(r + 1), (r + 1)(r + 2)$	r^2	$(r + 1)^2$	$(r^2 + r - 1)^2$
$(2, 2r(r + 1), 2(r + 1)(r + 2))$	$(2r + 1)^2$	$(2r + 3)^2$	$(2r^2 + 4r + 1)^2$
$(3, r(3r + 2), (r + 1)(3r + 5))$	$(3r + 1)^2$	$(3r + 4)^2$	$(3r^2 + 5r + 1)^2$
$(3, r(3r - 2), (r + 1)(3r + 1))$	$(3r - 1)^2$	$(3r + 2)^2$	$(3r^2 + r - 1)^2$
$(4, r(r + 1), (r + 2)(r + 3))$	$(2r + 1)^2$	$(2r + 5)^2$	$(r^2 + 3r + 1)^2$

We make two observations. The second and final examples in the list exploit the remarkable fact that the product of four consecutive integers plus 1 is a perfect square. The second observation is that, except for the first example, b and c are of the form $f(r)$ and $f(r + 1)$ for a quadratic polynomial and we see a relationship to the result in Vignette 4 on quadratic forms.

Two families of triples (p, q, r) for which $pq - 1$, $qr - 1$ and $rp - 1$ are all square are given by

$$(p, q, r) = (1, k^2 + 1, (k + 1)^2 + 1)$$

and

$$(p, q, r) = (2, (k - 1)^2 + k^2, k^2 + (k + 1)^2).$$

It turns out that if we have two numbers a and b for which $ab+1$ is a square, we can augment it by a third c so that $bc+1$ and $ac+1$ are squares. If $ab+1=m$, set $c=a+b+2m$. Then

$$ac+1=a^2+ab+2am+1=(a+m)^2$$

and

$$bc+1=ab+b^2+2bm+1=(b+m)^2.$$

Similarly, if $pq-1=n^2$ and $r=p+q+2n$, then $pr+1$ and $qr+1$ are both square.

In some cases at least, we can find infinitely many triples whose first two entries are given, with the help of Pell's equation. For example if $(1,3,u)$ is a triple for the case $k=1$, and $u+1=1\times u+1=y^2$, then we require that $3(y^2-1)+1=x^2$ for some x, or $x^2-3y^2=-2$. Some solutions of this are $(x,y)=(1,1),(5,3),(19,11)$ yielding the triples $(1,3,0)$, $(1,3,8)$, $(1,3,120)$. Likewise, if $(a,b,c)=(1,8,u)$ with $u=y^2-1$, we are lead to $x^2-8y^2=-7$ with solutions $(1,1)$, $(5,2)$, $(11,4)$ corresponding to $(a,b,c)=(1,8,0)$, $(1,8,3)$, $(1,8,15)$.

2. Sets of four integers. $(a,b,c,d)=(1,3,8,120)$ is a set of four integers for which the product of any pair plus 1 is a perfect square. This can be generalized to

$$(a,b,c,d)=(n-1,n+1,4n,4n(4n^2-1)$$

or

$$(1,n^2-1,n(n+2),4n(n^3+2n^2-1).$$

A two parameter family is given by

$$\begin{aligned}
&a=m\\
&b=n^2-1+(m-1)(n-1)^2\\
&c=n(mn+2)\\
&d=4m^3n^4+8m^2(2-m)n^3+4m(m-1)(m-5)n^2+4(2m-1)(m-2)n+4(m-1).
\end{aligned}$$

3. Families of sequences for different values of k. There are related infinite sequences for which any three consecutive entries a,b,c satisfy $ab+k$, $bc+k$ and $ca+k$ are all squares. Here is the first family with the squares for eahc triple given at the end:

$$\begin{array}{ccc}
k=-2 & (1,6,11,33,82,21,\cdots) & (2,3,8),(8,14,19),(19.30,52),(52,85,134)\\
k=-1 & (1,5,10,29,73,194.\cdots) & (2,3,7),(7,12,17),(17,27,46),(46,75,119)\\
k=0 & (1,4,9,25,64,169\cdots) & (2,3,6),(6,10,15),(15,24,40),(40,65,104)\\
k=1 & (1,3,8,21,55,144,377,\cdots) & (2,3,5),(5,8,13),(13,21,34),(34,55,89)\\
k=2 & (1,2,7,17,46,119,313,\cdots) & (2,3,4),(4,6,11),(11,18,28),(28,45,74)\\
k=3 & (1,1,6,13,37,94,\cdots) & (2,3,3),(3,4,9),(9,15,22),(22,35,59)\\
k=4 & (1,0,5,9,28,69,\cdots) & (2,3,2),(2,2,7),(7,12,16),(16,25,44)\\
k=5 & (1,-1,4,5,19,44,\cdots) & (2,3,1),(1,0,5),(5,9,10),(10,15,29)
\end{array}$$

Each sequence is obtained from its predecessor by subtracting the sequence $(0,1,1,4,9,25,64,\cdots)$ of squares of Fibonacci numbers. Hence the kth sequence is given by $g_{k,n}=f_n^2-kf_{n-2}^2$ and satisfies the recursion

$$g_{k,n+3}=2g_{k,n+2}+2g_{k,n+1}-g_{k,n}.$$

The second family is similar:

$$k=-2\quad(1,3,6,17,43,114,297,\cdots)\quad(1,2,4),(4,7,10),(10,16,27),(27,44,40)$$

$$
\begin{array}{lll}
k=-1 & (1,2,5,13,34,89,233,\cdots) & (1,2,3),(3,5,8),(8,13,21),(21,34,55) \\
k=0 & (1,1,4,9,25,64,169,\cdots) & (1,2,2),(2,3,6),(6,10,15),(15,24,40) \\
k=1 & (1,0,3,5,16,39,105,\cdots) & (1,2,1),(1,1,4),(4,7,9),(9,14,25) \\
k=2 & (1,-1,2,1,7,14,41,\cdots) & (1,2,0),(0,1,2),(2,4,3),(3,4,10)
\end{array}
$$

Each sequence is obtained from its predecessor by subtracting the sequence $(0,1,1,4,9,25,64,\cdots)$, so that the kth sequence is given by $h_{k,n} = f_n^2 - kf_{n-1}^2$ and satisfies the recursion

$$h_{k,n+3} = 2h_{k,n+2} + 2h_{k,n+1} - h_{k,n}.$$