Basic Algebra Test Solutions.

Answers to the Test:
1. T 2. F 3. F 4. T 5. T 6. F 7. C 8. A 9. B 10. D

Solutions and Comments:

1. $\sqrt[3]{-1} = -1$ is True.

   You can take the cube root of a negative number; you can take the cube root of *any* number. Some calculators don't accept negative arguments for their $\sqrt[x]{y}$ button, but that is a shortcoming of the way the calculator is programmed.

2. $\sqrt{a^2} = a$ is False.

   The $\sqrt{\ }$ symbol means the positive square root. (If you want both roots you have to write $\pm\sqrt{\ }$.) In this question, the statement $\sqrt{a^2} = a$ is false if $a < 0$. The statement that is always true is
   $$\sqrt{a^2} = |a|.$$

3. $\sqrt{a^2 + b^2} = a + b$ is False.

   There is no easy way to simplify $\sqrt{a^2 + b^2}$. Observe that
   $$\begin{aligned} \sqrt{a^2 + b^2} = a + b \quad &\Rightarrow \quad a^2 + b^2 = (a + b)^2 \\ &\Rightarrow \quad a^2 + b^2 = a^2 + 2ab + b^2 \\ &\Rightarrow \quad ab = 0 \end{aligned}$$

   So $\sqrt{a^2 + b^2} = a + b$ is true if both $a = 0$ and $b = 0$, or if one is zero and the other is positive. Consequently $\sqrt{a^2 + b^2} = a + b$ is *never* true if both $a, b \neq 0$.

4. $\sqrt{3^{2x} + 2 + 3^{-2x}} = 3^x + 3^{-x}$ is True.

   The expression inside the square root sign is a perfect square:
   $$a^2 + 2ab + b^2 = (a + b)^2,$$
   with $a = 3^x$ and $b = 3^{-x}$. Thus
   $$\begin{aligned} \sqrt{3^{2x} + 2 + 3^{-2x}} &= \sqrt{(3^x + 3^{-x})^2} \\ &= |3^x + 3^{-x}| \\ &= 3^x + 3^{-x}, \text{ since both } 3^x > 0 \text{ and } 3^{-x} > 0 \end{aligned}$$

5. If $a \neq 0, b \neq 0, a + b \neq 0$, then
   $$\frac{a + b}{\frac{1}{a} + \frac{1}{b}} = ab$$
   is True.

   Simplify left side:
   $$\frac{a + b}{\frac{1}{a} + \frac{1}{b}} = \frac{a + b}{\frac{b+a}{ab}} = (a + b)\frac{ab}{(a + b)} = ab$$

6. $|x|^3 = x^3$ is False.

The statement is true if $x \geq 0$, but not true if $x < 0$:

$$x < 0 \Rightarrow |x| = -x \Rightarrow |x|^3 = (-x)^3 = -x^3.$$

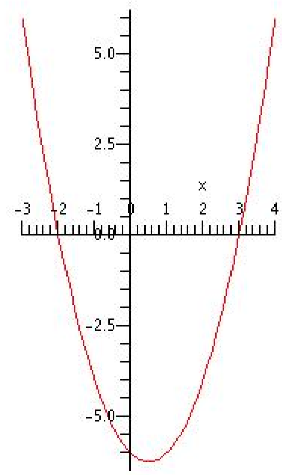
7. If $x^2 - x - 6 > 0$, then $x < -2$ or $x > 3$.

There are at least two ways to see this. Both require that you factor $x^2 - x - 6$.

Let

$$y = x^2 - x - 6 = (x-3)(x+2.)$$

The graph of y is a parabola opening upwards with x intercepts $x = -2$ and $x = 3$. So the graph of y is above the x-axis if $x < -2$ or $x > 3$.



Or you can solve the inequality by taking cases:

$$\begin{aligned} x^2 - x - 6 > 0 &\Leftrightarrow (x-3)(x+2) > 0 \\ &\Leftrightarrow x-3 > 0 \text{ and } x+2 > 0, \text{ or, } x-3 < 0 \text{ and } x+2 < 0 \\ &\Leftrightarrow x > 3 \text{ or } x < -2 \end{aligned}$$

8. The vertex of the parabola with equation $y = 5 + 6x - x^2$ is $(x, y) = (3, 14)$.

There are at least two ways to do this: one way using algebra, one way using calculus. You can complete the square:

$$y = 5 + 6x - x^2 = 14 - 9 + x - x^2 = 14 - (9 - x + x^2) = 14 - (x-3)^2;$$

so the vertex is $x = 3$ and $y = 14$.

Or you can find the vertex by setting $y' = 0$:

$$\frac{dy}{dx} = 6 - 2x = 0 \Leftrightarrow x = 3.$$

Then $y = 5 + 18 - 9 = 14$, as before.

9. The centre and radius of the circle with equation $x^2 + 2x + y^2 - 4y = 4$ are

$$\text{centre: } (x, y) = (-1, 2); \quad \text{radius: } r = 3.$$

This is an exercise in completing the square, both in the x variable, and in the y variable.

$$\begin{aligned} x^2 + 2x + y^2 - 4y = 4 \quad &\Leftrightarrow \quad x^2 + 2x + 1 + y^2 - 4y + 4 = 4 + 1 + 4 \\ &\Leftrightarrow \quad (x+1)^2 + (y-2)^2 = 9 \\ &\Leftrightarrow \quad (x+1)^2 + (y-2)^2 = 3^2 \end{aligned}$$

So the centre of the circle is $(x, y) = (-1, 2)$ and its radius is $r = 3$.

10. If $x < -1$, then

$$\sqrt{x^2 + x} + x = \frac{-1}{\sqrt{1 + 1/x} + 1}.$$

This is quite tricky. First of all, there are restrictions on x since

$$x^2 + x \geq 0 \Leftrightarrow x(x+1) \geq 0 \Leftrightarrow x \leq -1 \text{ or } x \geq 0.$$

The expression simplifies differently in each case. If $x > 0$, then the answer is actually B. In either case, you start by rationalizing the numerator:

$$\begin{aligned} \sqrt{x^2 + x} + x &= (\sqrt{x^2 + x} + x)\left(\frac{\sqrt{x^2 + x} - x}{\sqrt{x^2 + x} - x}\right) \\ &= \frac{x^2 + x - x^2}{\sqrt{x^2 + x} - x} \\ &= \frac{x}{\sqrt{x^2 + x} - x} \\ &= \frac{x}{\sqrt{x^2(1 + 1/x)} - x} \end{aligned}$$

If $x > 0$, then $\sqrt{x^2(x + 1/x)} = x\sqrt{1 + 1/x}$, and

$$\frac{x}{\sqrt{x^2(1 + 1/x)} - x} = \frac{x}{x\sqrt{1 + 1/x} - x} = \frac{1}{\sqrt{1 + 1/x} - 1};$$

but if $x < -1$, then $\sqrt{x^2} = |x| = -x$, so

$$\frac{x}{\sqrt{x^2(1 + 1/x)} - x} = \frac{x}{-x\sqrt{1 + 1/x} - x} = \frac{1}{-\sqrt{1 + 1/x} - 1} = \frac{-1}{\sqrt{1 + 1/x} + 1}.$$