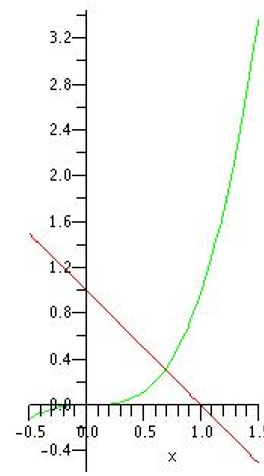
Functions Test Solutions.

Answers to the Test:
1. F 2. T 3. T 4. F 5. A 6. C 7. B 8. B 9. D
Solutions and Comments:

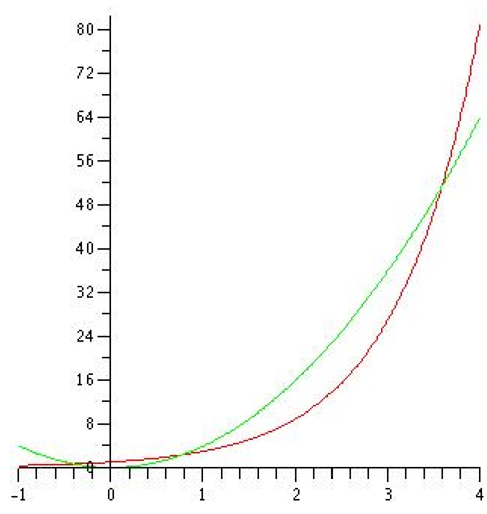
1. There are exactly three real solutions to the equation $x^3 = 1 - x$; this is False.

Any solution to this equation would represent an intersection point on the graphs of $y = x^3$ and $y = 1 - x$. The graphs are to the right. There is only one intersection point. So the solution to the equation $x^3 = 1 - x$ is somewhere between $x = 0$ and $x = 1$. Note: there *are* three solutions to the equation if you permit complex solutions.



2. There are exactly three real solutions to the equation $3^x = 4x^2$; this is True.

But it's not obvious. You can use the same approach as in the previous question. The graphs of $y = 3^x$ and $y = 4x^2$ are to the right: the quadratic is in green; the exponential is in red. You can see that there are three intersection points.



3. The range of the graph with equation $x^{2/3} + y^{2/3} = 4$ is $-8 \le y \le 8$; this is True.

$$0 \le x^{2/3} = 4 - y^{2/3} \Rightarrow y^{2/3} \le 4 \Rightarrow y^2 \le 64 \Rightarrow |y| \le 8 \Rightarrow -8 \le y \le 8.$$

4. If a sequence F_n is defined by $F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, then $F_6 = 6$; this is False.

   Aside: this is the Fibonacci sequence. Compute:

   $F_2 = F_1 + F_0 = 1; F_3 = F_2 + F_1 = 2; F_4 = F_3 + F_2 = 3; F_5 = F_4 + F_3 = 5; F_6 = F_5 + F_4 = 8.$

5. The inverse of the function $f(x) = \dfrac{2x+3}{x-5}$ is $f^{-1}(x) = \dfrac{5x+3}{x-2}$.

   To find the inverse of $y = f(x)$ interchange x and y and solve for y :

$$
\begin{aligned}
x = \frac{2y+3}{y-5} \quad &\Rightarrow \quad x(y-5) = 2y+3 \\
&\Rightarrow \quad xy - 5x = 2y + 3 \\
&\Rightarrow \quad xy - 2y = 5x + 3 \\
&\Rightarrow \quad y(x-2) = 5x + 3 \\
&\Rightarrow \quad y = \frac{5x+3}{x-2}
\end{aligned}
$$

6. The number of asymptotes to the graph of $f(x) = \dfrac{x^2+1}{x+1}$ is 2.

$$f(x) = \frac{x^2+1}{x+1} = x - 1 + \frac{2}{x+1},$$

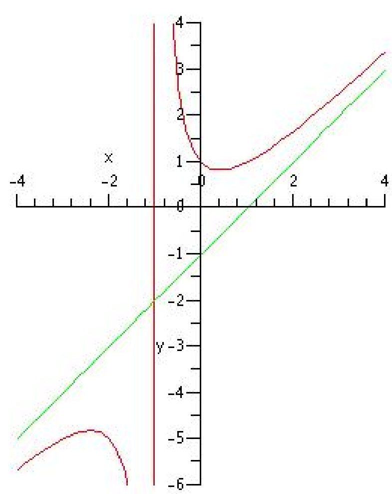
   by long division. So

$$x = -1$$

   is a vertical asymptote to the graph of f, and

$$y = x - 1$$

   is a slant asymptote to the graph of f. See the graph to the right.



7. If $g(x) = \dfrac{1}{x}$ and $h \neq 0$, then $\dfrac{g(x+h) - g(x)}{h} = \dfrac{-1}{x(x+h)}$.

Straight simplification:

$$\begin{aligned}\frac{g(x+h)-g(x)}{h} &= \frac{\frac{1}{x+h}-\frac{1}{x}}{h} \\ &= \frac{\frac{x-(x+h)}{x(x+h)}}{h} \\ &= \frac{-h}{hx(x+h)} \\ &= \frac{-1}{x(x+h)}\end{aligned}$$

8. Let $f(x) = 3x - 2$, let $g(x) = x^2 - 1$. Then $f(g(x)) = 3x^2 - 5$.

$$f(g(x)) = f(x^2 - 1) = 3(x^2 - 1) - 2 = 3x^2 - 3 - 2 = 3x^2 - 5.$$

9. If $|2x - 4| \leq |x + 3|$, then $\frac{1}{3} \leq x \leq 7$.

One way to solve this is to plot graphs. To the right, the red graph is the graph of

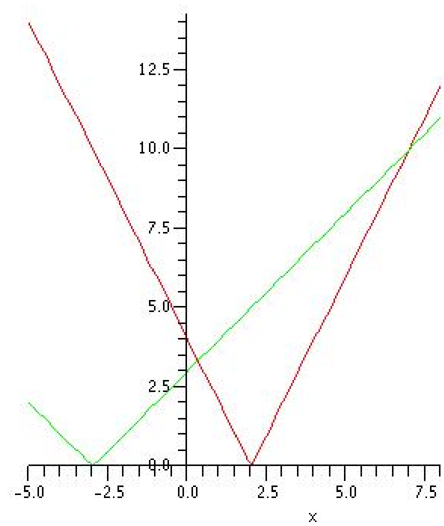
$$y = |2x - 4|$$

and the green graph is the graph of

$$y = |x + 3|.$$

You can see that the red graph is below the green graph for

$$\frac{1}{3} \leq x \leq 7.$$



Or you can solve the inequality algebraically, using $|z|^2 = z^2$ to eliminate the absolute value signs.

$$\begin{aligned}|2x-4| \leq |x+3| &\Leftrightarrow |2x-4|^2 \leq |x+3|^2 \\ &\Leftrightarrow (2x-4)^2 \leq (x+3)^2 \\ &\Leftrightarrow 4x^2 - 16x + 16 \leq x^2 + 6x + 9 \\ &\Leftrightarrow 3x^2 - 22x + 7 \leq 0 \\ &\Leftrightarrow (3x-1)(x-7) \leq 0 \\ &\Leftrightarrow \frac{1}{3} \leq x \leq 7\end{aligned}$$