FIRST YEAR MATH: General Advice August 20, 2019

Written Tests: All tests and exams require you to show your work and explain what you are doing. The final answer, even if it is correct, will be worth very little unless you have *explained* your solution. Moreover, final answers should always be simplified.

Partial credit can be obtained for partially correct work, but NO credit may be given if your work is poorly presented, difficult to decipher, uses incorrect mathematical notation, or makes no sense. It is not our responsibility to figure out what you have done; you are supposed to make it clear what you are doing. At

`https://www.math.utoronto.ca/burbulla/detailed`

you can find posted solutions to many old term tests and exams in MAT186/187/188. Consider these posted solutions as model solutions; they are what we expect you to do.

Homework: Lectures will expose you to new ideas and show you what kind of examples you should be able to do. But none of the material from lectures will really sink in until you try the homework. *To be successful in a math course you must do the homework.* You can get help with homework in the Math Aid Centre PG101, or from your lecturer during his or her office hours, or you can drop by the Math Aid Office, GB149.

It is advisable to do your homework on a regular (weekly) basis and to attend tutorials. Leaving things until the night before a quiz or a test is asking for trouble.

Notation: Mathematics has its own set of symbols. If you use them, you must use them correctly. (You will lose marks on tests or exams if your notation is incorrect.) Here are some examples of common errors:

Example 1: Suppose you are differentiating the function $y = x^2$. If you write

$$\begin{aligned} y &= x^2 \\ &= 2x \end{aligned}$$

it is incorrect, even though most math people would know what you are doing. Of course, you should write

$$\begin{aligned} y &= x^2 \\ \Rightarrow y' &= 2x \end{aligned}$$

Or you could simply write

$$\frac{dx^2}{dx} = 2x.$$

Example 2: Don't confuse equality with approximation: $\sqrt{2} \neq 1.414$; $\sqrt{2} \approx 1.414$

Example 3: Don't confuse implication ($\Rightarrow$) with equality ($=$) . The symbol $\Rightarrow$ is a logical connective and means If ... then ... For example,

$$\text{if } y = x^2 \text{ then } y' = 2x,$$

can be written as

$$y = x^2 \Rightarrow y' = 2x.$$

But writing something like

$$\begin{aligned} ax^2 + bx + c &\Rightarrow 0 \\ x &\Rightarrow \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

is abuse of notation.

Example 4: The following expressions

$$\frac{0}{0}, \frac{\infty}{\infty}, 0^0, 0 \cdot \infty, 1^\infty, \infty^0 \text{ and } \infty - \infty$$

are indeterminate. When evaluating limits of this type, which may or may not exist, don't *equate* the limit to one of the above expressions. For example, writing

$$\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1} = \frac{0}{0}$$

is incorrect; the limit is actually equal to $\frac{3}{2}$. You *can* say the limit is of the form $\frac{0}{0}$, or that when $x = 1$,

$$\frac{x^3 - 1}{x^2 - 1} = \frac{1 - 1}{1 - 1} = \frac{0}{0},$$

which is indeterminate.

Example 5: When reducing a matrix, don't put equal ($=$) signs between matrices which *aren't* equal:

$$\begin{pmatrix} 1 & 3 & -2 \\ -2 & 3 & 1 \\ 1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & -2 \\ 0 & 9 & -3 \\ 0 & -1 & 1 \end{pmatrix}$$

is incorrect. Instead, use an arrow ($\rightarrow$) or a tilde ($\sim$) between row equivalent matrices:

$$\begin{pmatrix} 1 & 3 & -2 \\ -2 & 3 & 1 \\ 1 & 2 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & -2 \\ 0 & 9 & -3 \\ 0 & -1 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 3 & -2 \\ -2 & 3 & 1 \\ 1 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -2 \\ 0 & 9 & -3 \\ 0 & -1 & 1 \end{pmatrix}.$$

Example 6: An indefinite integral always includes an arbitrary constant. To write

$$\int x^2 \, dx = \frac{x^3}{3}$$

is incomplete, and will cost you a mark everytime. Correct is:

$$\int x^2 \, dx = \frac{x^3}{3} + C.$$

Example 7: The correct indefinite integral of $\dfrac{1}{x}$ with respect to x is:

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

Without the absolute value it is wrong, and will cost you a mark everytime. Some students even write

$$\int \frac{1}{x}\,dx = \ln x,$$

which is doubly wrong – and will cost you *two* marks.

Logic and Proof: It seems as if many first year students have never been exposed to any formal logic in their high school curriculum. Unfortunately, logic is part and parcel of mathematics. You can't present a mathematical argument without using logic; you can't understand what a theorem says if you can't distinguish between hypotheses and conclusions. Here is some logical terminology you should become familiar with. Let P and Q be statements, mathematical or otherwise:

- Implication: $P \Rightarrow Q$ means "If P is true, then Q is true." Most theorems in mathematics are of this form. Consider the Intermediate Value Theorem:

  > Let f be a continuous function on the closed interval $[a, b]$ such that $f(a)f(b) < 0$. Then there is a number $c \in (a, b)$ such that $f(c) = 0$.

  P is the statement

  > Let f be a continuous function on the closed interval $[a, b]$ such that $f(a)f(b) < 0$.

  This is the *hypothesis* of the theorem. It is actually the conjunction of two statements: f is continuous on $[a, b]$ *and* $f(a)f(b) < 0$. These two conditions can also be called the hypotheses of the theorem.

  Q is the statement

  > There is a number $c \in (a, b)$ such that $f(c) = 0$.

  This is called the *conclusion* of the theorem.

  Warning: there are many different ways in English to say "$P \Rightarrow Q$." Here are some of them:

  1. Q is true if P is true.
  2. For Q to be true it is sufficient for P to be true.
  3. P is true only if Q is true.
  4. For P to be true it is necessary that Q is true.

- Negation: $\neg P$ represents the negation of P. The Law of the Excluded Middle states that every statement is true or its negation is true: one of P or $\neg P$ is always true. An application of this is proof by contradiction; that is, to show the statement A is true, assume that A is false and deduce a contradiction.

- Equivalent Statements: $P \Leftrightarrow Q$ means "P is true if and only if Q is true." Many important theorems in mathematics are of this form. For example, Let A be an $n \times n$ matrix. A is invertible if and only if $\det(A) \neq 0$.

P is the statement "A is invertible;" Q is the statement "$\det(A) \neq 0$." The theorem means that these two statements are equivalent; for given A, they are either both true or both false.

- Converse: the converse of $P \Rightarrow Q$ is $Q \Rightarrow P$. Thus P and Q are equivalent if both $P \Rightarrow Q$ *and* its converse $Q \Rightarrow P$ are true.
- Contrapositive: the contrapositive of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$. It is logically true that

$$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P).$$

That is, an implication and its contrapositive are equivalent statements.

Errors in logic that show up in your written solutions will cost you marks on a test or exam, and may make your whole "solution" worthless – even if your calculations are correct. For your interest, Appendix B in one of the textbooks for MAT188H1F, Nicholson's *Linear Algebra with Applications,* has a short, useful summary about basic logic and how to prove things in Mathematics. Appendix C describes Proof by Mathematical Induction, which may be used to prove some statements in MAT188H1F. In terms of your first year math courses you will be expected to understand proofs as given in class, and you are expected to be able to prove some statements, especially in MAT188H1F.

Example 8: The converse of a true statement is *not* necessarily a true statement. For example, it is a true statement that

If f is differentiable at x_0 then f is continuous at x_0.

The converse of this statement,

If f is continuous at x_0 then f is differentiable at x_0,

is false. For example, $f(x) = x^{1/3}$ is continuous at $x_0 = 0$, but it is not differentiable at $x_0 = 0$. Of course, the contrapositive of the original statement,

If f is not continuous at x_0 then f is not differentiable at x_0,

is also a true statement.

Example 9: To show a statement is *false* you only have to exhibit one counterexample. For example, the statement

If the number of equations in a system of linear equations is less than the number of unknowns in the system, then the system has infinitely many solutions.

is a false statement. Consider the example

$$\begin{array}{ccccccc} x & + & y & + & z & = & 0 \\ x & + & y & + & z & = & 1 \end{array}$$

which consists of two equations in three unknowns $(2 < 3)$ but has no solutions, because it is inconsistent.

Example 10: To show a statement is *true,* it is not enough to illustrate it with one example. Consider the statement

If A is a 2×2 matrix such that $A^2 = I$ then $\det(A) = \pm 1$.

which *is* a true statement. You can't prove it is true by simply considering one choice of A. Instead, you must give a general argument:

$$\begin{aligned} A^2 = I \quad &\Rightarrow \quad \det\left(A^2\right) = \det(I) \\ &\Rightarrow \quad (\det(A))^2 = 1 \\ &\Rightarrow \quad \det(A) = \pm 1 \end{aligned}$$

Inexcusable Algebraic Errors: Should any of the following types of algebraic errors show up in one of your solutions, everything after the appearance of the error will be forfeited. In computer jargon, these errors can all be considered "fatal errors."

Example 11: Both $\sqrt{a^2} = a$ and $\sqrt{a^2} = \pm a$ are wrong! Correct is:

$$\sqrt{a^2} = |a|.$$

Example 12: $\sqrt{a^2 + b^2} = a + b$ is wrong. There just isn't any easy way to simplify $\sqrt{a^2 + b^2}$.

Example 13: $\dfrac{1}{a+b} = \dfrac{1}{a} + \dfrac{1}{b}$ is wrong. Correct, for $a \neq 0, b \neq 0$ is

$$\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}.$$

So for $a + b \neq 0$,

$$\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b} \Leftrightarrow (a+b)^2 = ab \Leftrightarrow a^2 + ab + b^2 = 0,$$

which is *impossible* if a and b are non-zero real numbers.

Example 14: $\ln\left(\dfrac{M}{N}\right) = \dfrac{\ln M}{\ln N}$ and $\ln(MN) = \ln N \, \ln N$ are both wrong. Correct are

$$\ln\left(\frac{M}{N}\right) = \ln M - \ln N \text{ and } \ln(MN) = \ln M + \ln N,$$

if both $M > 0$ and $N > 0$.

Example 15: $\sin(\alpha+\beta) = \sin\alpha + \sin\beta$ and $\cos(\alpha+\beta) = \cos\alpha + \cos\beta$ are both wrong. Correct are:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

and

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Example 16: $\sin(k\theta) = k \sin\theta$ is wrong, unless $k = -1, 0, 1$. There is no easy way to simplify $\sin(k\theta)$, but here are two special cases

$$\sin(2\theta) = 2\sin\theta\cos\theta \text{ and } \sin(3\theta) = 3\sin\theta - 4\sin^3\theta.$$

Example 17: $\ln(M + N) = \ln M + \ln N$ is wrong. There just isn't any easy way to simplify $\ln(M + N)$.

Absolute Value: probably the most misunderstood formula from high school is the definition of absolute value:

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$$

That is, to simplify $|a|$ you have to take cases.

Example 18: Thus $|x|^2 = x^2$ and $\sqrt{x^2} = |x|$ are correct, but $|x|^3 = x^3$ is incorrect. Can you explain why?

Some Comments About Marking: If you make a copying error – or some other 'dumb' mistake – in your work which results in the rest of the problem being much easier than intended, then you will forfeit lots of marks. On the other hand, if you make a copying error and the rest of the problem remains of comparable difficulty, then you will only lose a few marks. Here is an example: consider the integral

$$\int \frac{1}{x^3 + 1}\, dx,$$

which is a fairly involved problem requiring partial fractions, completing the square and a trig substitution to solve. Say it is worth 10 marks on a test. If you miscopy the integral as

$$\int \frac{1}{x^3 - 1}\, dx$$

you could still get 9 out of 10, if the rest of your work is correct, because the exact same procedures as the intended question are involved. However, if you miscopy the question as

$$\int \frac{1}{x^2 + 1}\, dx$$

it becomes a one-liner, and you would get at most 1 mark out of 10.

Some Comments About Your Calculator: Don't blame your calculator if it gives you the wrong answer. For example, many calculators are programmed to accept only positive arguments when taking roots of a number. So, even though there is a real value for

$$(-1)^{(1/3)},$$

some caclulators will not evaluate this expression, or will display an answer of 0. The correct answer is of course

$$(-1)^{(1/3)} = -1.$$

Make sure you know how to use your calculator, and its built-in limits.

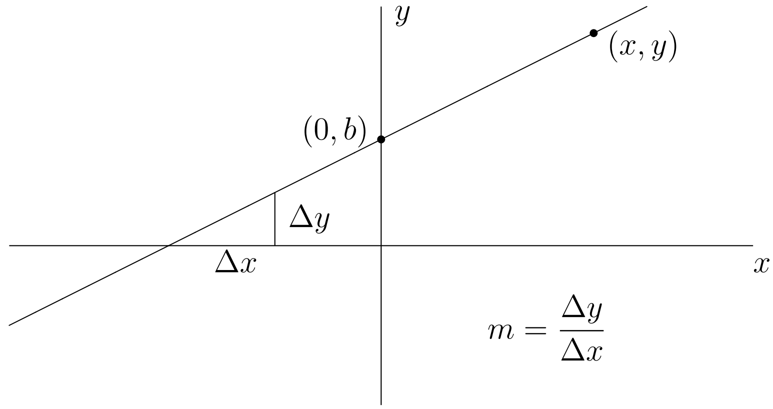
The same holds true for WolframAlpha and other mathematical software, like *Matlab, Maple* or *Mathematica*. They aren't always programmed as you might expect.

The following six pages list some formulas from high school with which you should be familiar.

Equations of Lines:
with slope m.

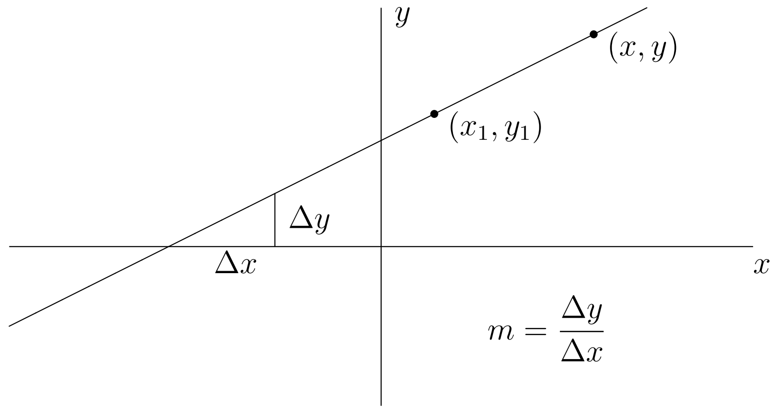
Slope, y-intercept form:

$$y = mx + b$$



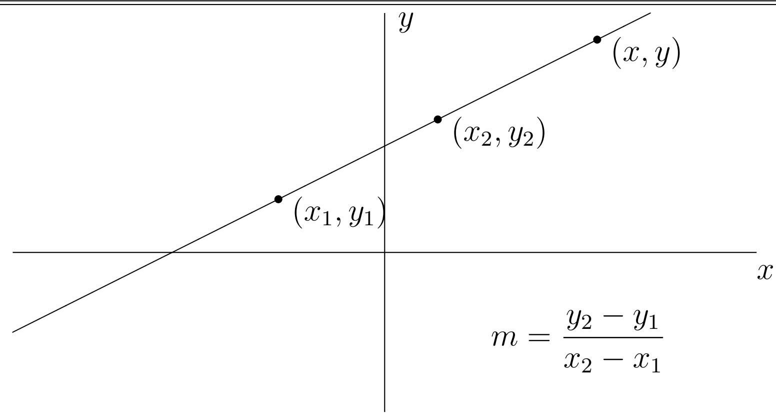
Slope-Point Form:

$$\frac{y - y_1}{x - x_1} = m$$



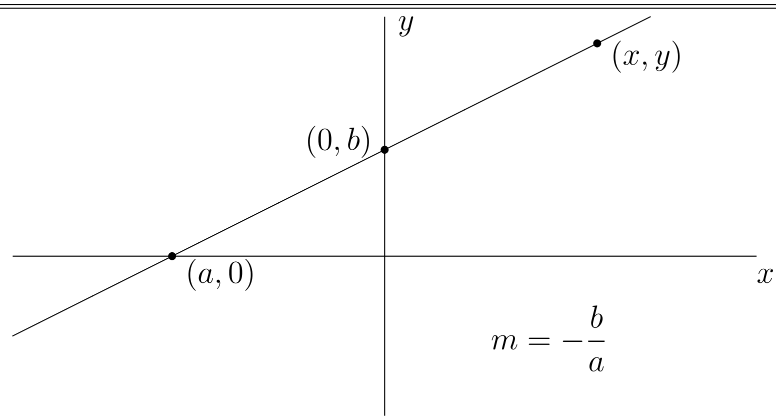
Two Points Form:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$



Intercept Form: for $ab \neq 0$.

$$\frac{x}{a} + \frac{y}{b} = 1$$



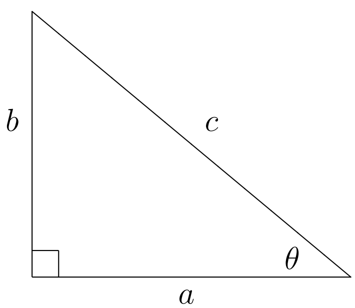
Quadratic Formula: suppose a, b, c are real numbers. Then

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The roots are

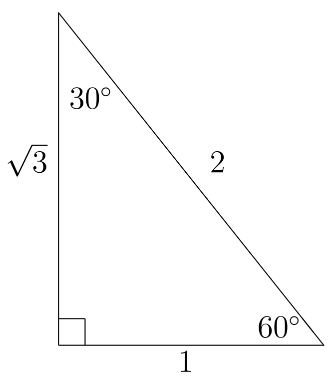
- real and distinct, if $b^2 - 4ac > 0$;
- real and repeated, if $b^2 - 4ac = 0$;
- complex conjugates, if $b^2 - 4ac < 0$.

The Six Trigonometric Functions and Right Triangles:



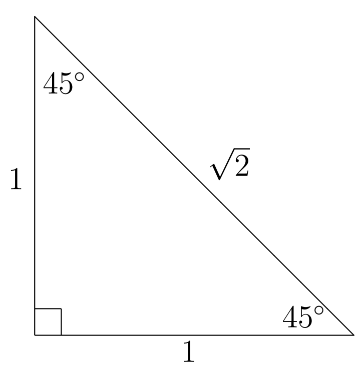
- $\sin\theta = \dfrac{b}{c}$
- $\cos\theta = \dfrac{a}{c}$
- $\tan\theta = \dfrac{b}{a} = \dfrac{\sin\theta}{\cos\theta}$

- $\csc\theta = \dfrac{c}{b} = \dfrac{1}{\sin\theta}$
- $\sec\theta = \dfrac{c}{a} = \dfrac{1}{\cos\theta}$
- $\cot\theta = \dfrac{a}{b} = \dfrac{1}{\tan\theta}$

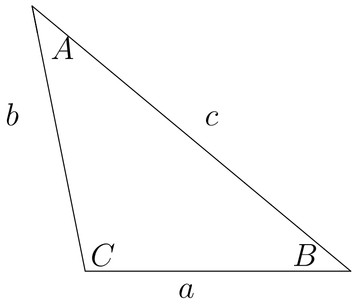


- $\sin 60^\circ = \dfrac{\sqrt{3}}{2}$
- $\cos 60^\circ = \dfrac{1}{2}$
- $\tan 60^\circ = \sqrt{3}$

- $\sin 30^\circ = \dfrac{1}{2}$
- $\cos 30^\circ = \dfrac{\sqrt{3}}{2}$
- $\tan 30^\circ = \dfrac{1}{\sqrt{3}}$



- $\sin 45^\circ = \dfrac{1}{\sqrt{2}}$
- $\cos 45^\circ = \dfrac{1}{\sqrt{2}}$
- $\tan 45^\circ = 1$



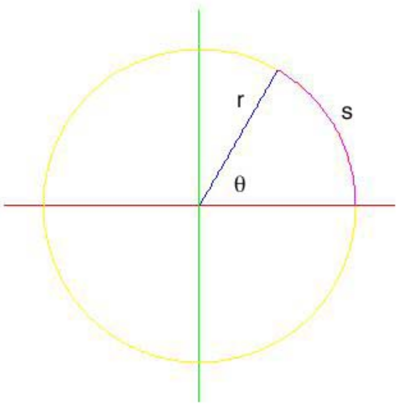
Sine Law:

- $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Cosine Law:

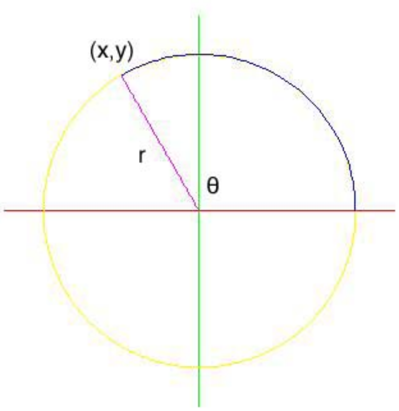
- $a^2 = b^2 + c^2 - 2bc\cos A$
- $b^2 = a^2 + c^2 - 2ac\cos B$
- $c^2 = a^2 + b^2 - 2ab\cos C$

Radian Measure and Circles:



- $\theta = \dfrac{s}{r}$
- $s = r\theta$
- π radians $= 180°$
- Radian measure is the ratio of two lengths; so it is unit free.

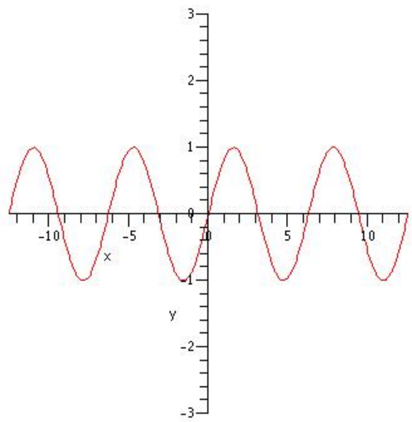
The Six Trigonometric Functions; θ in radians.



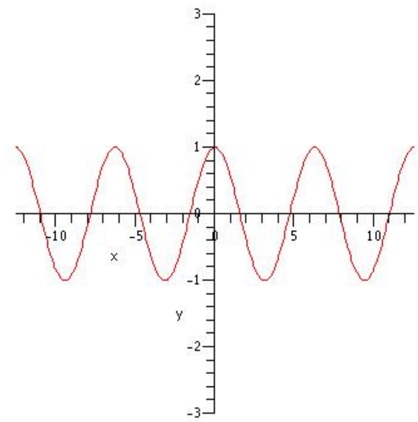
- $\sin\theta = \dfrac{y}{r}$
- $\cos\theta = \dfrac{x}{r}$
- $\tan\theta = \dfrac{y}{x}$
- $\csc\theta = \dfrac{r}{y}$
- $\sec\theta = \dfrac{r}{x}$
- $\cot\theta = \dfrac{x}{y}$

The Pythagorean Identities:

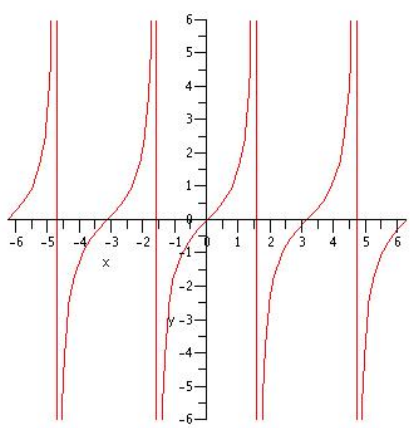
- $\sin^2\theta + \cos^2\theta = 1$
- $\tan^2\theta + 1 = \sec^2\theta$
- $1 + \cot^2\theta = \csc^2\theta$



Graph of $y = \sin x$.
Domain: $\mathbb{R}$; **Range:** $[-1, 1]$



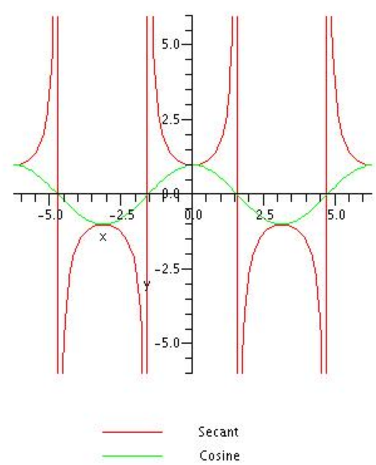
Graph of $y = \cos x$.
Domain: $\mathbb{R}$; **Range:** $[-1, 1]$



Graph of $y = \tan x$.

Domain: $x \neq \dfrac{(2k+1)}{2}\pi$;

Range: $\mathbb{R}$



Graph of $y = \sec x$.

Domain: $x \neq \dfrac{(2k+1)}{2}\pi$;

Range: $|y| \geq 1$

Negative angles:

- $\sin(-\theta) = -\sin\theta$
- $\cos(-\theta) = \cos\theta$
- $\tan(-\theta) = -\tan\theta$

Complementary angles:

- $\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta$
- $\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin\theta$
- $\tan\left(\dfrac{\pi}{2} - \theta\right) = \cot\theta$

Supplementary angles:

- $\sin(\pi - \theta) = \sin\theta$
- $\cos(\pi - \theta) = -\cos\theta$
- $\tan(\pi - \theta) = -\tan\theta$

Sum and Difference Formulas:

- $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$
- $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$
- $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
- $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Double Angle Formulas:

- $\sin(2\theta) = 2\sin\theta\cos\theta$
- $\cos(2\theta) = \cos^2\theta - \sin^2\theta$
- $\cos(2\theta) = 1 - 2\sin^2\theta$
- $\cos(2\theta) = 2\cos^2\theta - 1$

Reformulation of Double Angle Formulas:

$$\bullet \sin^2\theta = \frac{1-\cos(2\theta)}{2} \qquad \bullet \cos^2\theta = \frac{1+\cos(2\theta)}{2};$$

$$\bullet \sin^2\frac{A}{2} = \frac{1-\cos A}{2} \qquad \bullet \cos^2\frac{A}{2} = \frac{1+\cos A}{2}.$$

Properties of Exponentials: Let a, b be positive numbers; let x, y be real numbers. Then

$$\bullet\ a^{x+y} = a^x a^y \qquad \bullet\ a^{x-y} = \frac{a^x}{a^y} \qquad \bullet\ a^{xy} = (a^x)^y \qquad \bullet\ (ab)^x = a^x b^x$$

$$\bullet\ a^0 = 1 \qquad \bullet\ a^1 = a \qquad \bullet\ a^{-1} = \frac{1}{a}.$$

Properties of Logarithms, for $a > 0, a \neq 1$: Let x and y be positive real numbers; let z be any real number.Then:

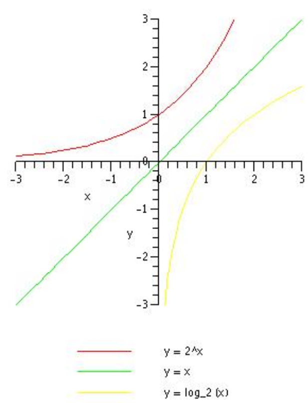
$$\bullet \log_a(xy) = \log_a x + \log_a y \qquad \bullet \log_a 1 = 0$$

$$\bullet \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y \qquad \bullet \log_a a = 1$$

$$\bullet \log_a(x^z) = z\log_a x \qquad \bullet \log_a\left(\frac{1}{a}\right) = -1$$

$$\bullet \log_a a^z = z \qquad \bullet\ a^{\log_a x} = x$$

Graphs of $y = 2^x$ and $y = \log_2 x$



Definition of e, the Natural Base of Exponentials and Logarithms:

- $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2.71828$

Derivatives:

- $\frac{dc}{dx} = 0$
- $\frac{dx^n}{dx} = nx^{n-1}$
- $\frac{d\sin x}{dx} = \cos x$
- $\frac{d\cos x}{dx} = -\sin x$
- $\frac{de^x}{dx} = e^x$
- $\frac{d\ln x}{dx} = \frac{1}{x}$

Differentiation Rules:

- $\frac{d(f(x)+g(x))}{dx} = \frac{df(x)}{dx} + \frac{dg(x)}{dx}$
- $\frac{d(f(x)-g(x))}{dx} = \frac{df(x)}{dx} - \frac{dg(x)}{dx}$
- $\frac{d\,(c\,f(x))}{dx} = c\frac{df(x)}{dx}$
- $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
- $\frac{d(u/v)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
- $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

Note: probably most of these formulas will be reviewed in your MAT186H1F lectures, time permitting. However, you are expected to understand them and be able to use them, when necessary, even if they aren't covered in your lectures.