Problem: (Fourier method to study resonance)

$$u_{tt} - 4u_{xx} = \cos^2 x \sin \omega t, \quad t > 0, \quad 0 < x < \pi$$

$$u_x(0,t) = u_x(\pi, t) = 0 \qquad t > 0$$

$$u(x,0) = 0, \quad u_t(x,0) = 0 \qquad 0 < x < \pi$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$ ✓ (Student was right :-))

$$u(x,t) = C_0(t) + \sum_{n=1}^{\infty} C_n(t) \cos nx$$ (Neumann boundary conditions)

$$C_0'' + \sum_{n=1}^{\infty} C_n'' \cos nx - 4\sum_{n=1}^{\infty} C_n(-n^2)\cos nx = \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)\sin \omega t$$

$n=0$: $\quad C_0'' = \frac{1}{2}\sin \omega t$

$n=2$: $\quad C_2'' + 16\,C_2 = \frac{1}{2}\sin \omega t$

$n \neq 0$ and $n \neq 2$: $\quad C_n'' + 4n^2 C_n = 0 \Rightarrow C_n = a\cos 2nt + b\sin 2nt$

$\Rightarrow C_n' = -2na\sin 2nt + 2nb\cos 2nt$

+ initial values $C_n(0) = 0$ because

$$\begin{cases} u(x,0) = C_0(0) + \sum_{n=1}^{\infty} C_n(0)\cos nx = 0 \Rightarrow C_n(0) = 0 \\ u_t(x,0) = C_0'(0) + \sum_{n=1}^{\infty} C_n'(0)\cos nx = 0 \Rightarrow C_n'(0) = 0 \end{cases}$$

$C_n(0) = a \Rightarrow a = 0, \qquad C_n'(0) = 2nb = 0 \underset{n \neq 0}{\Rightarrow} b = 0$

$\Rightarrow$ if $n \neq 0$ and $n \neq 2 \Rightarrow C_n(t) = 0$

$\boxed{n=0;}$ Case $\omega \neq 0$:

$$C_0'' = \frac{1}{2}\sin\omega t$$

$$C_0 = a + bt + c\sin\omega t$$

$$C_0' = b + c\omega\cos\omega t$$

$$C_0'' = -c\omega^2\sin\omega t$$

$$-c\omega^2\sin\omega t = \frac{1}{2}\sin\omega t$$

$$-c\omega^2 = \frac{1}{2} \Rightarrow c = -\frac{1}{2\omega^2}$$

if $\boxed{\omega \neq 0} \Rightarrow C_0 = a + bt - \frac{1}{2\omega^2}\sin\omega t$

$$C_0' = b - \frac{1}{2\omega}\cos\omega t$$

$$C_0(0) = 0 = a \Rightarrow a = 0;$$

$$C_0'(0) = b - \frac{1}{2\omega} \Rightarrow b = \frac{1}{2\omega};$$

$$C_0(t) = \frac{1}{2\omega}t - \frac{1}{2\omega^2}\sin\omega t;$$

if $\boxed{\omega = 0} \Rightarrow C_0'' = 0 \Rightarrow C_0 = a + bt$

$$C_0(0) = a = 0, \quad C_0'(0) = b = 0 \Rightarrow C_0 = 0$$

---

$\boxed{n=2}$ Case $\boxed{\omega \neq \pm 4:}$

$$C_2'' + 16C_2 = \frac{1}{2}\sin\omega t$$

$$C_2 = a\cos 4t + b\sin 4t + c\sin\omega t + d\cos\omega t$$

$$C_2' = -4a\sin 4t + 4b\cos 4t + c\omega\cos\omega t - d\omega\sin\omega t$$

$$C_2'' = -16a\cos 4t - 16b\sin 4t - c\omega^2\sin\omega t - d\omega^2\cos\omega t;$$

$$C_2'' + 16C_2 = -c\omega^2\sin\omega t - d\omega^2\cos\omega t + 16c\sin\omega t = \frac{1}{2}\sin\omega t \Rightarrow c = \frac{1}{2(16-\omega^2)}; \quad d = 0$$

$$C_2 = a\cos 4t + b\sin 4t + \frac{1}{2(16-\omega^2)}\sin\omega t$$

$$C_2' = -4a\sin 4t + 4b\cos 4t + \frac{\omega}{2(16-\omega^2)}\cos\omega t$$

$$C_2(0) = a = 0 \Rightarrow a = 0$$

$$C_2'(0) = 4b + \frac{\omega}{2(16-\omega^2)} \Rightarrow b = \frac{\omega}{8(\omega^2-16)}$$

if $\boxed{\omega \neq \pm 4} \Rightarrow \boxed{C_2(t) = \frac{\omega}{8(\omega^2-16)}\sin 4t + \frac{1}{2(16-\omega^2)}\sin\omega t}$

(if $\omega = 0 \Rightarrow C_2(t) = 0$ just check)

if $\boxed{\omega = 4}$

$$C_2'' + 16C_2 = \frac{1}{2}\sin 4t$$

$$C_2 = a\cos 4t + b\sin 4t + ct\cos 4t + dt\sin 4t$$

$$C_2' = -4a\sin 4t + 4b\cos 4t - 4ct\sin 4t + c\cos 4t + 4dt\cos 4t + d\sin 4t$$

$$C_2'' = -16a\cos 4t - 16b\sin 4t - 16ct\cos 4t - 4c\sin 4t - 16dt\sin 4t + 4d\cos 4t + 4d\cos 4t$$

$$C_2'' + 16C_2 = -4c\sin 4t + 8d\cos 4t = \frac{1}{2}\sin 4t$$

$$d = 0; \quad c = -\frac{1}{8} \Rightarrow$$

$$C_2(t) = a\cos 4t + b\sin 4t - \frac{1}{8} t\cos 4t;$$

$$C_2'(t) = -4a\sin 4t + 4b\cos 4t - \frac{1}{8}\cos 4t + \frac{4}{8} t\sin 4t$$

$$C_2(0) = 0 = a \Rightarrow a = 0$$

$$C_2'(0) = 0 = 4b - \frac{1}{8} \Rightarrow b = \frac{1}{32};$$

$$C_2(t) = \frac{1}{32}\sin 4t - \frac{1}{8} t\cos 4t;$$

if $\omega = -4 \Rightarrow$
$\tilde{C}_2(t) = -C_2(t)$

---

if $\boxed{\omega = 0} \Rightarrow u(x,t) \equiv 0$ ✓

if $\boxed{\omega \neq 0 \text{ and } \omega \neq \pm 4} \Rightarrow$

3 different types of the solution !!!

$$u(x,t) = \frac{1}{2\omega} t - \frac{1}{2\omega^2}\sin\omega t + \left(\frac{\omega}{8(\omega^2-16)}\sin 4t + \frac{\sin\omega t}{2(16-\omega^2)}\right)\cos 2x$$

if $\boxed{\omega = \pm 4}$

$$u(x,t) = \frac{1}{2\omega} t - \frac{1}{2\omega^2}\sin\omega t \pm \left(\frac{1}{32}\sin 4t - \frac{1}{8} t\cos 4t\right)\cos 2x$$