Solve by 2 methods!

$u_t = u_{xx} + u$

$u(0,t) = u(\pi,t) = 0$

$u(x,0) = 1$

Separation of variable method:

$u(x,t) = X(x)\,T(t)$

$$\left.\begin{aligned} u(0,t) &= X(0)\,T(t) = 0 \\ u(\pi,t) &= X(\pi)\,T(t) = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} X(0) &= 0 \\ X(\pi) &= 0 \end{aligned}$$

$$XT' = X''T + XT$$

$$\frac{T'}{T} = \frac{X''}{X} + 1 = \lambda$$

$$\begin{cases} X'' = (\lambda - 1)\,X \\ X(0) = X(\pi) = 0 \end{cases}$$

case 1:
$\lambda - 1 = 0$ – no eigenvalues

case 2:
$\lambda - 1 > 0$ – no eigenvalues

case 3: $\lambda - 1 < 0$

$$X = a\sin\left(\sqrt{-\lambda+1}\,x\right) + b\cos\left(\sqrt{-\lambda+1}\,x\right)$$

$X(0) = 0 \Rightarrow b = 0$

$X(\pi) = 0 \Rightarrow \sqrt{-\lambda+1} = n \Rightarrow -\lambda + 1 = n^2 \Rightarrow$

$\Rightarrow \lambda_n = -n^2 + 1$, $n = 1, 2, 3, 4\ldots$ eigenvalues

$X_n = \sin(nx)$ – eigenfunctions.

$$\frac{T_n'}{T_n} = -n^2 + 1, \quad n = 1, 2, 3, 4\ldots$$

$$T_n = e^{(-n^2+1)t} \Rightarrow u(x,t) = \sum_{n=1}^{+\infty} c_n e^{(1-n^2)t} \sin(nx)$$

$$u(x,0) = \sum_{n=1}^{+\infty} c_n \sin(nx) = 1 \Rightarrow c_n = \frac{(1, \sin(nx))}{(\sin(nx), \sin(nx))} =$$

$$= \frac{2}{\pi n}\left(1 - (-1)^n\right);$$

Fourier method:

$u_t = u_{xx} + u$

$u(0,t) = u(\pi, t) = 0$

$u(x,0) = 1$

$$\begin{cases} Lf = f'' + f \\ f(0) = f(\pi) = 0 \end{cases} \text{ self-adjoint in } L^2(0,\pi)$$

eigenvalue problem: $Lf = \lambda f$

$$\left.\begin{array}{l} f'' + f = \lambda f \\ f(0) = f(\pi) = 0 \end{array}\right\} \Rightarrow \begin{array}{l} \lambda_n = -n^2 + 1, \quad n = 1, 2, 3, 4 \ldots \\ f_n = \sin(nx) \end{array}$$

(solved on the first page)

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t)\, f_n = \sum_{n=1}^{\infty} c_n(t) \sin(nx)$$ ← using basis representation

$$u_t = u_{xx} + u \Rightarrow \sum_{n=1}^{\infty} c_n'(t) \sin(nx) = \sum_{n=1}^{\infty} (-n^2)\, c_n(t) \sin(nx) +$$

$$+ \sum_{n=1}^{\infty} c_n(t) \sin(nx); \quad c_n'(t) = (1-n^2)\, c_n(t) \Rightarrow c_n(t) = e^{(1-n^2)t}$$

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{(1-n^2)t} \sin(nx)$$

$$u(x,0) = 1 = \sum_{n=1}^{\infty} a_n \sin(nx) \Rightarrow a_n = \frac{(1, \sin nx)}{(\sin nx, \sin nx)} =$$

$$= \frac{2}{\pi n}\left(1 - (-1)^n\right)$$

Both methods lead to the solution:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{\pi n}\left(1 - (-1)^n\right) e^{(1-n^2)t} \sin(nx)$$