Homework 10.
Solutions and marking scheme.

Total:
$4+6+10=20$

6.1.5

$u_{xx}+u_{yy}=1$, PDE; $x^2+y^2<a$

$u\big|_{x^2+y^2=1}=0$; IV;
$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$

$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=1.$;

Assume:
$u(r,\theta)\equiv u(r)$; $\quad u_{rr}+\frac{1}{r}u_r=1 \Rightarrow (ru_r)'=r \Rightarrow$

$\Rightarrow$

$ru_r=\frac{r^2}{2}+C \Rightarrow u=\frac{r^2}{4}+c_1\ln r+c_2$; (2)

$c_1=0$ since $\ln r\to\infty$ if $r\to 0$.

$u=\frac{r^2}{4}+c_2$; $\quad u(a)=\frac{a^2}{4}+C=0$ from IV

$u(x,y)=\frac{x^2+y^2}{4}-\frac{a^2}{4}$; (2)

Total 4.

6.3.3

$u_{xx}+u_{yy}=0$, $\quad r<a$

$u=\sin^3\theta=\frac{3}{4}\sin\theta-\frac{\sin 3\theta}{4}$;

General solution:

$$u(r,\theta)=\frac{A_0}{2}+\sum_{n=1}^{\infty}r^n\left(A_n\cos(n\theta)+B_n\sin(n\theta)\right) \quad (2)$$

$$\frac{3}{4}\sin\theta-\frac{\sin 3\theta}{4}=\frac{A_0}{2}+\sum_{n=1}^{\infty}a^n\left(A_n\cos(n\theta)+B_n\sin(n\theta)\right) \quad (2)$$

$A_n=0$ because $\sin^3\theta$ is an odd function, $B_n=0$ if $n\neq 1$, $n\neq 3$

$u(r,\theta)=\frac{3r}{4a}\sin\theta-\frac{r^3}{4a^3}\sin 3\theta$; (2)

Total 6

6.2.4.

Method 1.

$$u_{xx} + u_{yy} = 0, \quad (0 < x < 1,\ 0 < y < 1)$$
$$u(x,0) = x, \quad u(x,1) = 0$$
$$u_x(0,y) = 0, \quad u_x(1,y) = y^2.$$

$$V(x,y) = u(x,y) - \frac{x^2}{2}y^2 \quad (\text{Shift method})$$
$$V_x = u_x - xy^2$$
$$V_x(0,y) = u_x(0,y) - 0\cdot y^2 = 0;$$
$$V_x(1,y) = u_x(1,y) - 1\cdot y^2 = y^2 - y^2 = 0;$$
$$V(x,0) = u(x,0) - \frac{x^2}{2}\cdot 0 = u(x,0) = x;$$
$$V(x,1) = u(x,1) - \frac{x^2}{2}\cdot 1 = 0 - \frac{x^2}{2} = -\frac{x^2}{2};$$
$$V_{xx} = u_{xx} - y^2;$$
$$V_y = u_y - x^2 y;$$
$$V_{yy} = u_{yy} - x^2;$$

$$V_{xx} + V_{yy} = -(x^2 + y^2);$$
$$V_x(0,y) = V_x(1,y) = 0; \quad \sim \text{Neumann B.C.}$$
$$V(x,0) = x, \quad V(x,1) = -\frac{x^2}{2};$$

Can be solved for $V(x,y)$ by standart Fourier method.

Method 2. $(\Delta u = u_{xx} + u_{yy})$

$$\Delta u_1 = 0 \qquad\qquad \Delta u_2 = 0$$
$$u_1(x,0) = x, \quad u_1(x,1) = 0, \ \text{Dirichlet BC} \qquad u_2(x,0) = u_2(x,1) = 0$$
$$u_{1x}(0,y) = 0, \quad u_{1x}(1,y) = 0; \ \text{Neumann BC} \qquad u_{2x}(0,y) = 0, \quad u_{2x}(1,y) = y^2;$$

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$$u(x,y) = u_1(x,y) + u_2(x,y).$$

Both problems for $u_1(x,y)$ and $u_2(x,y)$ can be solved by standart Fourier method.

$$u_1(x,y) = \frac{A_0}{2}(y-1) + \sum_{n=1}^{\infty} A_n \left(e^{n\pi y} - e^{2n\pi} e^{-n\pi y}\right)\cos(n\pi x)$$

$u_1(x,0) = x \Rightarrow$

$$\Rightarrow x = -\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \left[1 - e^{2n\pi}\right] \cos(n\pi x) = x$$

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$$-\frac{A_0}{2} = \int_0^1 x \cos(n\pi x)\, dx = \frac{1}{2}; \Rightarrow A_0 = -1;$$

$$A_n\left(1 - e^{2n\pi}\right) = 2\int_0^1 x\cos(n\pi x)\,dx =$$

$$= \frac{2}{n^2\pi^2}\left((-1)^n - 1\right) \Rightarrow A_n = \frac{2}{n^2\pi^2\left(1 - e^{2\pi n}\right)}\left((-1)^n - 1\right);$$

$$u_2(x,y) = \sum_{n=1}^{\infty} B_n \left(e^{n\pi x} + e^{-n\pi x}\right)\sin(n\pi y);$$

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$$u_{2x}(1,y) = \sum_{n=1}^{\infty} B_n\, n\pi\left(e^{n\pi} - e^{-n\pi}\right)\sin(n\pi y) = y^2;$$

$$B_n = \frac{1}{n\pi\left(e^{n\pi} - e^{-n\pi}\right)}\left(\frac{2(-1)^{n+1}}{n\pi} + \frac{4}{n^3\pi^3}\left[(-1)^n - 1\right]\right);$$

$$u(x,y) = u_1(x,y) + u_2(x,y).$$

Total:

$4+4+2 = 10$