Question 1.

a). $u_x = 0 \Rightarrow u = f(y)$

b). $u_{xy} = 0 \Rightarrow u = f(y) + g(x)$

c). $u_{xx} - 4u_{yy} = 0 \Rightarrow u = f(y-2x) + g(y+2x)$

d). $u_x - 2u_y = 0 \Rightarrow u = f(2x+y)$

Question 2.

$u(x,y) = x^2y + y^3x$

$du = u_x\,dx + u_y\,dy = (2yx + y^3)\,dx + (x^2 + 3y^2x)\,dy$

$du\big|_{(1,2)} = (4+8)\,dx + (1+12)\,dy = 12\,dx + 13\,dy$

Question. 3.

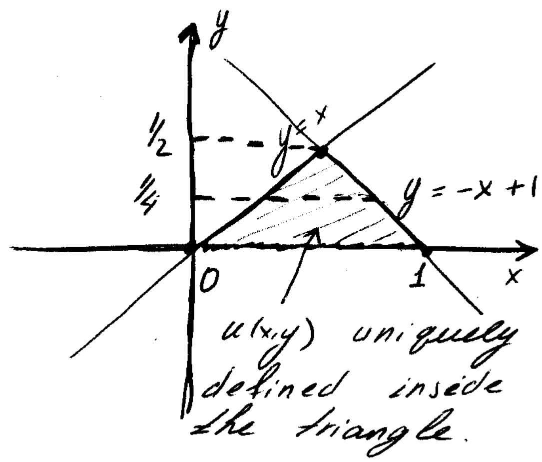
$u_{xx} - u_{yy} = 0$,

$u(x,0) = \varphi(x)$, $u_y(x,0) = \psi(x)$, $0 \le x \le 1$

$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)(u_x + u_y) = 0$

characteristics lines $y = x + c_1$

$y = -x + c_2$



a) $y=0$, $0 \le x \le 1$

b). $y = \frac{1}{4}$, $\frac{1}{4} \le x \le \frac{3}{4}$

c). $y = \frac{1}{2}$, $x = \frac{1}{2}$

d). $y = 1$, $x \in \emptyset$ (no points where $u(x,y)$ is uniquely defined).

Question 4.

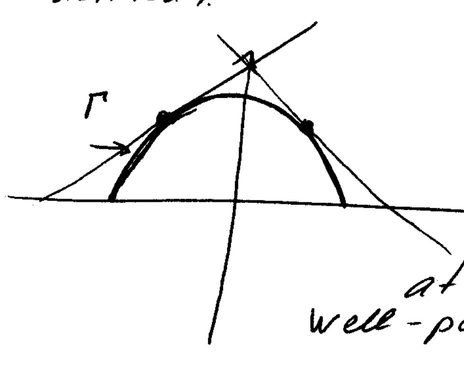
$u_{xx} - u_{yy} = 0$

$\Gamma = \{x^2 + y^2 = 1,\ y > 0\}$

characteristics lines

$y = x + c_1$

$y = x - c_1$



At two points $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ characteristics are tangent to Γ $\Rightarrow$ at these points local well-posedness does not hold.

$u = e^{\alpha x} v;$

$u_x = e^{\alpha x}(\alpha v + v_x)$

$u_{xx} = e^{\alpha x}(\alpha^2 v + 2\alpha v_x + v_{xx})$

$u_{yy} = e^{\alpha x} v_{yy}$

$u_{xx} - 4u_{yy} + 2u_x + u = 0 \quad \Rightarrow$

$e^{\alpha x} v_{xx} - 4e^{\alpha x} v_{yy} + e^{\alpha x}(2\alpha + 2) v_x + e^{\alpha x}(\alpha^2 + 2\alpha + 1) = 0$

Take $\alpha = -1;$

$u = e^{-x} v; \qquad u(x,0) = x$

$v = e^x u \quad \Rightarrow \quad v(x,0) = e^x x$

$u_y = e^{-x} v_y; \quad u_y(x,0) = 0 \quad \Rightarrow \quad v_y(x,0) = 0$

PDE $\rightarrow$ $v_{xx} - 4v_{yy} = 0 \quad \Rightarrow \quad v_{yy} = \frac{1}{4} v_{xx}$

IV $\rightarrow$ $v(x,0) = e^x x$

$v_y(x,0) = 0$

d'Alembert solution: $v(x,y) = \frac{1}{2}\left((x - \frac{y}{2}) e^{x - \frac{y}{2}} + (x + \frac{y}{2}) e^{x + \frac{y}{2}} \right)$

$$\boxed{u(x,y) = \frac{1}{2}\left((x - \frac{y}{2}) e^{-\frac{y}{2}} + (x + \frac{y}{2}) e^{\frac{y}{2}} \right)}$$

What if I do not remember d'Alembert solution.

$v = f(y - 2x) + g(y + 2x)$ } general sol. of PDE

$v_y = f'(y - 2x) + g'(y + 2x) \quad \Rightarrow \quad f(-2x) + g(2x) = xe^x; \quad f'(-2x) + g'(2x) = 0;$

$\Rightarrow -2f'(-2x) + 2g'(2x) = (xe^x)' \quad \Rightarrow \quad \begin{cases} 4g'(2x) = (e^x x)'; \\ -4f'(-2x) = (e^x x)'; \end{cases}$

$g(2x) = \frac{1}{2x} e^x; \quad f(-2x) = \frac{1}{2x} e^x;$

$g(x) = x e^{\frac{x}{2}}; \quad f(x) = -x e^{-\frac{x}{2}};$

$v = \frac{1}{4}\left((2x - y) e^{x - \frac{y}{2}} + (2x + y) \cdot e^{x + \frac{y}{2}} \right) \quad \Rightarrow \quad \boxed{u(x,y) = \frac{1}{2}\left((x - \frac{y}{2}) e^{-\frac{y}{2}} + (x + \frac{y}{2}) e^{\frac{y}{2}} \right)}$

Problem 2. (Fourier method).

$y u_y = u_{xx} + 9u$, $u(0,y) = u(\pi,y) = 0$, $y>0$

a) $u(x,y)$? if $u(x,0) = 0$.

$y u_y - 9u = u_{xx}$

Solve eigenvalue problem:

$$\begin{cases} f'' = \lambda f \\ f(0) = f(\pi) = 0 \end{cases}$$

(Standart Dirichlet problem).

case 1: $\lambda = 0$ - no solutions

case 2: $\lambda > 0$ - no solutions

case 3: $\lambda < 0$

$\lambda_n = -n^2$, $n = 1, 2, 3 \ldots$

$f_n = \sin(nx)$

Use basis: $u(x,y) = \sum_{n=1}^{\infty} C_n(y) \sin(nx)$

$$\sum_{n=1}^{\infty} y\, C_n' \sin(nx) - 9 \sum_{n=1}^{\infty} C_n \sin(nx) = \sum_{n=1}^{\infty} (-n^2)\, C_n \sin(nx)$$

$$y\, C_n' = (9 - n^2)\, C_n$$

$$y \cdot \frac{dC_n}{dy} = (9-n^2)\, C_n \Rightarrow \frac{dC_n}{C_n} = (9-n^2)\frac{dy}{y} \Rightarrow$$

$$\Rightarrow \ln|C_n| = (9-n^2) \ln|y| \Rightarrow \boxed{C_n(y) = A_n y^{9-n^2}}$$

$u(x,y) = \sum_{n=1}^{\infty} A_n y^{9-n^2} \sin(nx)$, A_n are arb. constants

$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y)$; $u_n = A_n y^{9-n^2} \sin(nx)$

$u_1(x,0) = 0$ and $u_2(x,0) = 0$

$$\boxed{u(x,y) = a\, u_1(x,y) + b\, u_2(x,y)}$$

b). solution is not unique = not well-posed.

Problem 2 (Separation of variables)

a). $y u_y - 9u = u_{xx}$, $u(x,y) = X(x)Y(y)$

$yXY' - 9XY = X''Y$

$\frac{yY'}{Y} - 9 = \frac{X''}{X} = \lambda$; $u(0,y) = X(0)Y(y) = 0 \Rightarrow X(0) = 0$

$u(\pi,y) = X(\pi)Y(y) = 0 \Rightarrow X(\pi) = 0$

Solve eigenvalue problem:

$$\left\{\begin{array}{l} X'' = \lambda X \\ X(0) = X(\pi) = 0 \end{array}\right\} \Rightarrow \begin{array}{l} \boxed{\lambda_n = -n^2} \text{ - eigenvalues} \\ \boxed{X_n = \sin(nx)} \text{ - eigenfunctions} \end{array}$$

$\frac{yY_n'}{Y_n} - 9 = -n^2 \Rightarrow \frac{yY_n'}{Y_n} = 9-n^2 \Rightarrow \frac{dY_n}{Y_n} = (9-n^2)\frac{dy}{y}$

$Y_n = A_n y^{9-n^2} \Rightarrow u(x,y) = \sum_{n=1}^{\infty} A_n y^{9-n^2} \sin(nx) = \sum_{n=1}^{\infty} u_n(x,y)$

$u(x,0) = 0 \Rightarrow \boxed{u(x,y) = a y^8 \sin(nx) + b y^5 \sin(nx)}$

only $u_1(x,0) = 0$, $u_2(x,0) = 0$.

b). Solution is not unique $u_1(x,y) = y^8 \sin(nx)$,

$u_2(x,y) = y^5 \sin(nx) \Rightarrow$ problem is not well-posed.