Homework 3 (Solutions and marking scheme)

Problem 1. (solve by 2 methods)

$u_x + 2xu_y = 2xu \qquad u(0,y) = y^2$

$\Gamma: x = 0$

Method 1: $U(s) = u(x(s), y(s))$;

$\frac{dx}{ds} = 1; \quad x(0) = 0$ ① $\qquad x(s) = s$ ①

$\frac{dy}{ds} = 2x; \quad y(0) = \alpha$ ① $\qquad y(s) = s^2 + \alpha$ ①

$\frac{dU}{ds} = 2xU; \quad U(0) = \alpha^2$ ① $\qquad U(s) = \alpha^2 e^{s^2}$ ①

Family of characteristic curves:

$y = x^2 + \alpha$ ①

$u(x,y) = (y - x^2)^2 e^{x^2}$ ①

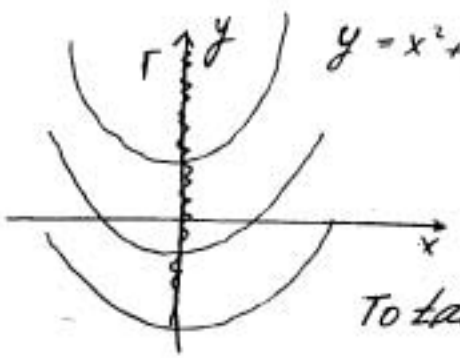
Method 2: $U(x) = u(x, y(x))$

$\frac{dy}{dx} = 2x; \quad y(0) = \alpha$ ① $\qquad y = x^2 + \alpha$ ①

$\frac{dU}{dx} = 2xU; \quad U(0) = \alpha^2$ ① $\qquad U = \alpha^2 e^{x^2}$ ①

$u(x,y) = (y - x^2)^2 e^{x^2}$ ①

IVP is well-posed in the entire (x,y) plane. ①



Total: 14

1.6.1. (b)

$9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$

$a_{11}a_{22} - a_{12}^2 = 9 \cdot 1 - 3^2 = 0 \Rightarrow$ this is a parabolic equation (1)

Total: 1

1.6.2 Find the region in the xy-plane where the eqn. $(1+x)u_{xx} + 2xy\,u_{xy} - y^2 u_{yy} = 0$ is elliptic, hyperbolic, or parabolic. Sketch.

$a_{11}a_{22} - a_{12}^2 = -(1+x)y^2 - x^2y^2 = -y^2(x^2+x+1) = -y^2\left((x+\frac{1}{2})^2 + \frac{3}{4}\right)$

I. $-y^2\left((x+\frac{1}{2})^2 + \frac{3}{4}\right) > 0 \Rightarrow -y^2 > 0 \Rightarrow$ the eqn. is never elliptic. (1)

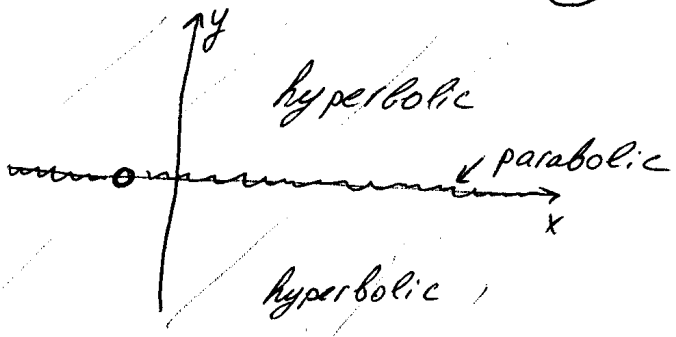
II. $-y^2\left((x+\frac{1}{2})^2 + \frac{3}{4}\right) < 0 \Rightarrow -y^2 < 0 \Rightarrow y \neq 0$

$\Rightarrow$ if $y \neq 0$ then the eqn. is hyperbolic. (1)

I. $-y^2\left((x+\frac{1}{2})^2 + \frac{3}{4}\right) = 0 \Rightarrow y^2 = 0 \Rightarrow y = 0$

at the point $x = -1$, $y = 0$ the eqn. is a degenerate one because all coef. are 0.

$\Rightarrow$ if $y = 0$ and $\boxed{x \neq -1}$ then the eqn. is parabolic. (2)



Total: 4

$\boxed{14 + 1 + 4 = 19}$