Homework 2 (Solutions and marking scheme)

1.2.2

$$3u_y + u_{xy} = 0, \quad v = u_y$$

! $u(x,y)$ and $v(x,y)$ are both functions of two variables.

$$3v + v_x = 0 \quad (\text{this is PDE!})$$

① $\boxed{-\frac{dv}{v} = 3\,dx} \Rightarrow -\ln|v| = 3x + C_1(y)$

$\boxed{v(x,y) = C_2(y)e^{-3x}}$ ①

$u_y = C_2(y)e^{-3x} \Rightarrow \boxed{u(x,y) = C_3(y)e^{-3x} + C_4(x)}$ ①

Total 3

1.2.5

$$\begin{cases} \sqrt{1-x^2}\,u_x + u_y = 0 \\ u(0,y) = y \end{cases}$$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \Rightarrow \boxed{y = \arcsin x + c}$ ①

$u(x,y) = f(y - \arcsin x)$, $u(0,y) = f(y - \arcsin(0)) =$

$= f(y) = y \Rightarrow \boxed{u(x,y) = y - \arcsin x}$ ①

Total 2

1.5.1

(ODE) $u'' + u = 0$

characteristic eqn. $k^2 + 1 = 0$, $k_{1,2} = \pm i$

general solution: $\boxed{u = C_1 \sin x + C_2 \cos x}$ ①

$u(0) = C_1 \cdot 0 + C_2 \cdot 1 = C_2 = 0 \Rightarrow$

$u(x) = C_1 \sin x$

$u(L) = C_1 \sin(L) = 0 \Rightarrow C_1 = 0$ or $L = n\pi$, $n = 0, \pm 1 \ldots$

The uniqueness of the solution depends on L. ①

If $\boxed{L \neq n\pi \Rightarrow C_1 = 0 \Rightarrow u(x) \equiv 0}$ is the unique solution. (both arbitrary constants C_1 and C_2 from the general solutio can be defined uniquely) ①

If $\boxed{L = n\pi \Rightarrow u(x) = C_1 \sin(x)}$ - infinitely many solutions because C_1 is not uniquely defined.

Total 3

1.5.3

$u'' = 0$ for $0 < x < 1$

$u' = C_1 \Rightarrow u = C_1 x + C_2$ is a general solution of $u'' = 0$. $u' = C_1$;

case I

$$\left.\begin{aligned} u'(0) + k\,u(0) &= C_1 + k(C_1 \cdot 0 + C_2) = C_1 + kC_2 = 0 \\ u'(1) + k\,u(1) &= C_1 + k(C_1 \cdot 1 + C_2) = (k+1)C_1 + kC_2 = 0 \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{array}{l} kC_1 = 0 \Rightarrow C_1 = 0 \text{ if } k \neq 0. \\ \text{if } k = 0 \Rightarrow C_1 + 0 \cdot C_2 = 0 \Rightarrow C_1 = 0. \end{array}\right\} \Rightarrow$$

$$\boxed{C_1 = 0 \text{ for } \forall\, k.}$$

If $C_1 = 0 \Rightarrow kC_2 = 0 \Rightarrow \boxed{k = 0 \text{ or } C_2 = 0}$

$\Rightarrow \boxed{u(x) \equiv 0 \text{ if } k \neq 0,\ C_2 = 0}$ 1 (unique)

$\boxed{u(x) = C_2 \text{ if } k = 0.}$ 1 (nonunique)

Case II

$$\left.\begin{array}{l} u'(0) + k\,u(0) = C_1 + kC_2 = 0 \\ u'(1) - k\,u(1) = C_1(1-k) - kC_2 = 0 \end{array}\right\} \Rightarrow$$

$C_1(2-k) = 0 \Rightarrow k = 2$ or $C_1 = 0$

If $C_1 = 0 \Rightarrow kC_2 = 0 \Rightarrow k = 0$ or $C_2 = 0$

If $k = 2 \Rightarrow C_1 + 2C_2 = 0 \Rightarrow C_1 = -2C_2$

If $k = 0 \Rightarrow C_1 = 0$ and C_2 is arbitrary.

I. If $k = 0 \Rightarrow u(x) = C_2$ 1 (non unique)

II. If $k = 2 \Rightarrow u(x) = -2C_2 x + C_2 = C_2(1-2x)$ 1 (non unique)

III. If $k \neq 0$ and $k \neq 2 \Rightarrow u(x) \equiv 0$ 1 (unique)

Total 5

$3+2+3+5 = 13$

(no half grades)