Question 2.

$$2u_{xx} - 8u_{xy} + 8u_{yy} + u_y = 0$$

a). $2 \cdot 8 - (-4)^2 = 0$, parabolic

b). $u_{xx} - 4u_{xy} + 4u_{yy} = \left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial y}\right)$

$\frac{dy}{dx} = -2 \Rightarrow y = -2x + c \Rightarrow \boxed{\xi = y + 2x, \quad \eta = x}$

any choice of a linearly independent variable

$J = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

c). $\frac{\partial u}{\partial x} = 2u_\xi + u_\eta$; $\frac{\partial u}{\partial y} = u_\xi$; $\frac{\partial^2 u}{\partial x \partial y} = 2u_{\xi\xi} + u_{\xi\eta}$;

$\frac{\partial^2 u}{\partial x^2} = 4u_{\xi\xi} + 4u_{\xi\eta} + u_{\eta\eta}$; $\frac{\partial^2 u}{\partial y^2} = u_{\xi\xi}$;

$$2(4u_{\xi\xi} + 4u_{\xi\eta} + u_{\eta\eta}) - 8(2u_{\xi\xi} + u_{\xi\eta}) +$$
$$+ 8u_{\xi\xi} + u_\xi = 2u_{\eta\eta} + u_\xi = 0$$

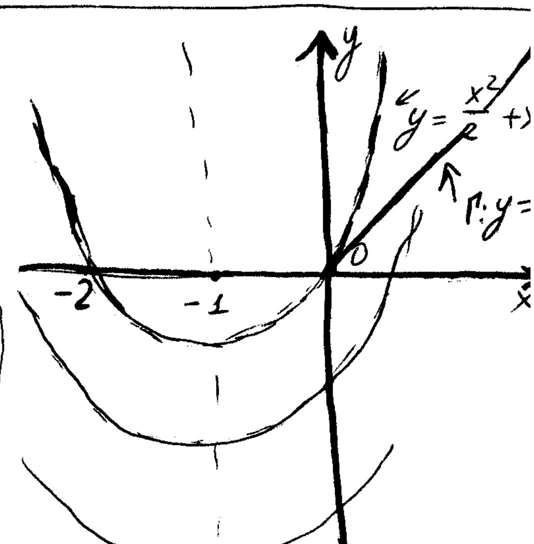
$\boxed{u_{\eta\eta} + \frac{1}{2}u_\xi = 0}$ (many alternative correct solutions)

Question 3.

$$u_x + (x+1)u_y = 0$$

a). $\frac{dy}{dx} = x + 1 \Rightarrow \boxed{y = \frac{x^2}{2} + x + C}$

b). $c = y - \frac{x^2}{2} - x \Rightarrow \boxed{u(x,y) = f\left(y - \frac{x^2}{2} - x\right)}$



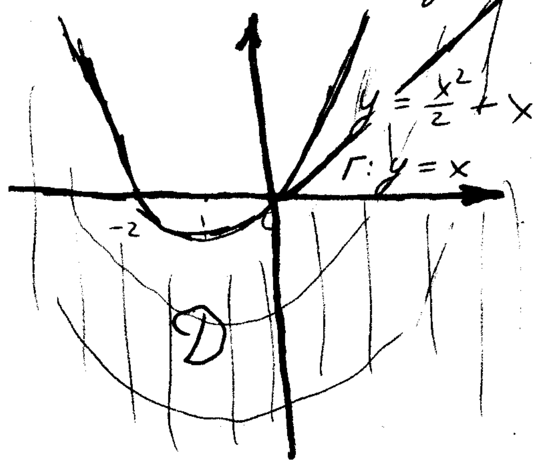
c). $u(x,x) = f\left(-\frac{x^2}{2}\right) = \sin(x)$, $x \geq 0$

$\Gamma: y = x$, $x \geq 0$

$\tau = -\frac{x^2}{2} \Rightarrow x = \sqrt{-2\tau} \Rightarrow \boxed{u(x,y) = \sin\left(\sqrt{-2\left(y - \frac{x^2}{2} - x\right)}\right)}$

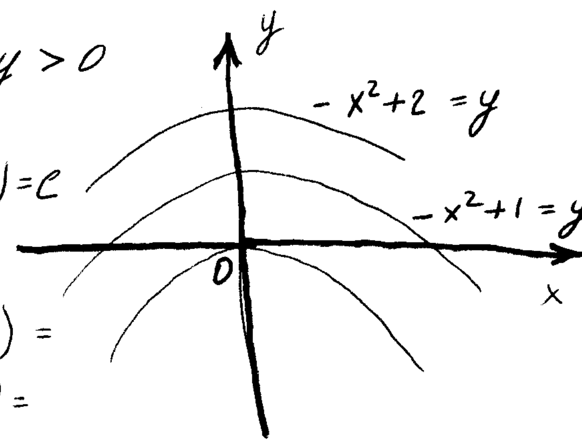
d). at $x=0, y=0$ the initial curve Γ: $y=x$ is tangent to the characteristic line $y=\frac{x^2}{2}+x$. There is no a local well-posedness at $(0,0)$.

e). The solution exists and unique if $y<\frac{x^2}{2}+x$.



Question 4.

$$u_x - 2x u_y = 2\frac{x}{y}u, \quad x>0,\ y>0$$



a). $\frac{dy}{dx}=-2x \Rightarrow \boxed{y=-x^2+c} \Rightarrow y(0)=c$

b). $\mathcal{U}(x)=u(x,y(x)) \Rightarrow \mathcal{U}(0)=u(0,y(0))=$
$=u(0,c)=f(c)=$
$=f(y+x^2)$

$$\frac{d\mathcal{U}}{dx}=\frac{2x}{y}\mathcal{U} \Rightarrow \frac{d\mathcal{U}}{\mathcal{U}}=\frac{2x}{c-x^2}dx \Rightarrow \ln|\mathcal{U}|=-\ln|c-x^2|+C_1 \Rightarrow$$

$$\Rightarrow \mathcal{U}=C_2\cdot\frac{1}{c-x^2}=C_2\cdot\frac{1}{y};\ \mathcal{U}(0)=C_2\cdot\frac{1}{y(0)}=C_2\cdot\frac{1}{c}=$$

$$=C_2\cdot\frac{1}{y+x^2}=f(y+x^2) \Rightarrow C_2=g(y+x^2) \Rightarrow$$

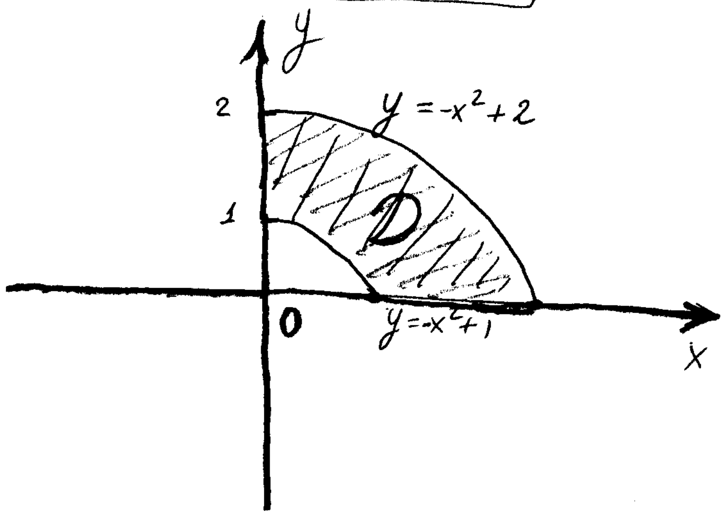
$$\Rightarrow \boxed{u(x,y)=g(y+x^2)\frac{1}{y}}$$

c). $u(0,y) = y^3, \quad 1 \le y \le 2$

$u(0,y) = g(y) \cdot \frac{1}{y} = y^3 \Rightarrow g(y) = y^4 \Rightarrow$

$\boxed{u(x,y) = (y+x^2)^4 \cdot \frac{1}{y}}$ $\Rightarrow y=0$ is a singular point

d).



The area between $y=-x^2+2$ and $y=-x^2+1$ is the domain where the solution exists and unique.