Infinite Domain Problems ($-\infty < x, y < +\infty$):

Problem 0

Find the region in xy plane where the equation

$$(1-x)u_{xx} + 2\,y\,u_{xy} - (1+x)u_{yy} = 0$$

is elliptic, hyperbolic, parabolic.

Problem 1

Solve by any method:

$$u_{yy} - u_{yx} - 2\,u_{xx} = 0, \qquad u(x,2x) = \phi(x), \qquad u_y(x,2x) = \psi(x)$$

Answer:

$$-2/3(\phi(y-x)) + 5/3(\phi((x+2y)/5)) + 5/3\int_0^{y-x}\psi(\tau)d\tau - \int_0^{(x+2y)/5}\psi(\tau)d\tau$$

Problem 2

Solve by any method:

$$u_{yy} - 4u_{xx} = \sin(x-y), \qquad u(x,0) = u_y(x,0) = 0$$

Answer:

$$u(x,y) = 1/4sin(2y-x) - 1/12sin(2y+x) + 1/3sin(x-y)$$

Problem 3

Solve using Green's formula:

$$u_{yy} - u_{xx} = \cos(x+y), \qquad u(x,0) = \cos(x), \qquad u_y(x,0) = \sin(x)$$

Check your answer by direct substitution.

Problem 4

Use the energy method to prove that if

$$u_{xx} - 4\,u_{yy} = 0, \qquad u(x,0) = u_y(x,0) = 0, \quad 0 \le x \le l$$

then $u(x,y) \equiv 0$ inside the triangle domain of dependence
with the base $[0,l]$.

Similar to one solved in class (with coefficient 1 instead of 4)

Problem 5 (challenging problem)

Use the energy method to prove that for t big enough

$$\frac{1}{2}\int_{-\infty}^{+\infty}(u_t)^2 dx = \frac{1}{2}\int_{-\infty}^{+\infty}(u_x)^2 dx$$

(equipartition of kinetic and potential energies for large times) if

$$u_{xx} - u_{tt} = 0, \qquad u(x,0) = \phi(x), \qquad u_t(x,0) = \psi(x)$$

where $\psi(x)$ and $\phi(x)$ have compact support.

Curious how to solve ask me or my TAs.

Finite Domain Problems:

Problem 6

Solve:

$$u_{tt} - u_{xx} + u = 0, \qquad u(0,t) = u(\pi,t) = 0, \qquad 0 < x < \pi$$

Answer:

$$u(x,t) = \sum_{n=1}^{+\infty} (a_n \cos(\sqrt{(n^2+1)}t) + b_n \sin(\sqrt{(n^2+1)}t)) sin(nx)$$

See for two more solutions in the fourier.pdf

Problem 7

Solve:

$$u_{tt} - 4u_{xx} = 0, \qquad u_x(0,t) = u_x(\pi,t) = 0, \qquad 0 < x < \pi, \quad u(x,0) = 0, \quad u_t(x,0) = 1$$

Check your answer by direct substitution.

Problem 8

Solve:

$$u_t = u_{xx}, \qquad u_x(0,t) = u_x(2\pi,t) = 0, \qquad 0 < x < 2\,\pi, \qquad u(x,0) = \cos(x)$$

Check your answer by direct substitution.

Problem 9

Solve:

$$u_t = 4u_{xx}, \qquad u(0,t) = u(2\pi,t) = 0, \qquad 0 < x < 2\,\pi, \qquad u(x,0) = e^x$$

Check your answer by direct substitution.