Real Analysis Qualifying Exam
September 2015

Please be brief but justify your answers, citing relevant theorems. Sometimes a sketch can help!

1. (a) In a couple of sentences, describe Lebesgue measure on $\mathbb{R}^d$.
   (You need not provide a construction, but aim for an unambiguous characterization.)

   (b) Assume that $E \subset \mathbb{R}^d$ is compact, and set, for $\varepsilon > 0$,

   $$U_\varepsilon = \left\{ x \in \mathbb{R}^d : \operatorname{dist}(x, E) < \varepsilon \right\} .$$

   Prove that their Lebesgue measure satisfies

   $$\lim_{\varepsilon \to 0^+} m(U_\varepsilon) = m(E) .$$

   (c) Show by example that the conclusion from (b) can fail for bounded open sets.

2. (a) Let $(f_n)_{n\geq 1}$ be a sequence of nonnegative measurable functions converging pointwise a.e. to an integrable function f. If $\lim \int f_n = \int f$, show that

   $$\int_E f = \lim_{n\to\infty} \int_E f_n$$

   for all measurable sets E.

   (b) Let $(g_n)_{n\geq 1}$ be a sequence of Lebesgue measurable functions on the interval $[0, 1]$ converging pointwise a.e. to some function g.
   If $|g_n(x)| < |x|^{-\frac{1}{3}}$, show that

   $$(*) \qquad \lim_{n\to\infty} \int_0^1 g_n(x)h(x)\,dx = \int_0^1 g(x)h(x)\,dx$$

   for all $h \in L^2[0, 1]$.

   (c) If, instead, $||g_n||_{L^2} \leq M$ for all n, show that $(*)$ holds for all bounded functions h.
   (*Hint:* Egoroff's theorem.)

3. Let f, g be complex-valued functions on $\mathbb{R}^d$.

(a) Define the Fourier transform $\mathcal{F} : f \mapsto \hat{f}$ for $f \in L^1$. Likewise for $f \in L^2$.
What can you say about $||\hat{f}||_\infty$ (for $f \in L^1$) and $||\hat{f}||_2$ (for $f \in L^2$)?

(b) Let $f * g$ denote the convolution of two integrable functions f and g, given by

$$f * g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,dy\,.$$

Express the Fourier transform of $f * g$ in terms of $\hat{f}$ and $\hat{g}$. (Please provide a full proof.)
Is your formula valid when $f, g \in L^2$? In what sense?

(c) Show that the Fourier transform defines a bounded linear operator from L^p to its dual, and provide an estimate on its operator norm. (*Hint:* Interpolate between $p = 1$ and $p = 2$.)

4. Let $\mathcal{H}$ be an infinite-dimensional Hilbert space (such as $L^2(\mathbb{R})$).

(a) What does it mean for a sequence $(v_n)_{n\geq 1}$ to be *orthonormal*?

(b) Define *weak convergence* $(f_n \rightharpoonup f)$ in $\mathcal{H}$.
Show that every orthonormal sequence converges weakly to zero.

(c) What is the precise relationship between *bounded* sequences and weak convergence in $\mathcal{H}$? Please state two relevant theorems.

(d) Given a vector $f \in \mathcal{H}$ with norm $||f|| < 1$, construct a sequence of unit vectors that converges weakly to f, that is, find $(u_n)_{n\geq 1}$ with

$$||u_n|| = 1 \quad (\text{for all } n \geq 1), \qquad \text{and} \qquad u_n \rightharpoonup f \quad (n \to \infty)\,.$$

Hint: Consider the intersection of the unit sphere $\{g \in \mathcal{H} : \ ||g|| = 1\}$ with the subspace $f^\perp$.