Department of Mathematics
University of Toronto

Topology Comprehensive Exam

September 19, 2016

Topology I

Problem 1. Let M^n, N^m be smooth manifolds such that $m > n$. Let $f\colon M^n \to N^m$ be a smooth map.

Prove that f is not $1-1$.

Hint: Use the constant rank theorem.

Problem 2. Let $n \geq 1$ and let M^n be a closed (i.e compact and with no boundary) oriented manifold. Let $f\colon M \to \mathbb{S}^n$ be a smooth map such that $\int_M f^*\omega \neq 0$ for some $\omega \in \Omega^n(S^n)$.

Prove that f is onto.

Problem 3. Let M be a compact manifold without boundary. Let V_1, V_2 be smooth vector fields on M transverse to the zero section of TM.

Prove that the number of zeros of V_1 has the same parity as the number of zeros of V_2.

Topology II

Problem 4. Show that if a path-connected locally path-connected space X has $\pi_1(X)$ finite, then each map $X \longrightarrow S^1$ is nullhomotopic.

Problem 5. Assume that $f : S^n \longrightarrow S^n$ is *not* homotopic to the antipodal map (that sends each x to its antipodal point $-x$). Prove that f has a fixed point. (In other words, there exists x such that $f(x) = x$.)

Problem 6. Let M^{2n} be a closed orientable manifold of dimension $2n$ such that $H_{n-1}(M^{2n}; \mathbb{Z})$ is torsion-free. Prove that $H_n(M^{2n}; \mathbb{Z})$ is also torsion-free.