Department of Mathematics
University of Toronto

Topology Comprehensive Exam

September 11, 2015, 10am–1pm

Topology I

Problem 1. Let $\mathbb{S}^3 = \{(z_0, z_1) \in \mathbb{C}^2 : |z_0|^2 + |z_1|^2 = 1\}$. Consider the map $\pi\colon \mathbb{S}^3 \to \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ given by $\pi(z_0, z_2) = (2z_0\bar{z}_1, |z_0|^2 - |z_1|^2)$.

(1) Show that $\pi(\mathbb{S}^3) \subset \mathbb{S}^2 \subset \mathbb{R}^3$.
(2) Show that $\pi\colon \mathbb{S}^3 \to \mathbb{S}^2$ is a submersion.
Hint: You can use without proof that that for any $p, q \in \mathbb{S}^3$ there are $A \in U(2), B \in O(3)$ such that $Ap = q$ and $\pi(Ap) = B\pi(q)$.

Problem 2. Consider the following subset of $\mathbb{R}^3$:

$$C = \{(x, y, z) : x^3 = y^2 + z^2\}$$

Is C a smooth submanifold of $\mathbb{R}^3$? Justify your answer.

Problem 3. Prove that the unit ball $B(0, 1) = \{x \in \mathbb{R}^n : |x| < 1\}$ is diffeomorphic to $\mathbb{R}^n$ for any $n \geq 1$.

Topology II

Problem 4. Use the Mayer-Vietoris sequence to find the homology groups of spheres $\mathbb{S}^n, n \geq 1$.

Hint: You may assume $H_1(\mathbb{S}^1) = \mathbb{Z}$.

Problem 5. Use Van Kampen's theorem to compute the fundamental group of the figure eight.

Hint: You can use the fact that the fundamental group of $\mathbb{S}^1$ is $\mathbb{Z}$.

Problem 6. Find the universal cover of
$$\mathbb{S}^2 \setminus \{N, S\}$$
where N and S are the north and the south pole respectively.