DEPARTMENT OF MATHEMATICS
University of Toronto

Analysis Comprehensive Exam

September 28, 2020

Time: 3 hours. Please be brief but justify your work.
If you make a reference to a textbook result, be sure to carefully quote it (correctly!).

1. State ...

   (a) ... Fubini's Theorem;

   (b) ... the Lebesgue Differentiation Theorem;

   (c) ... Hölder's Inequality;

   (d) ... the Open Mapping Theorem.

   Remember to give the assumptions as well as the conclusions!

2. Consider a sequence $\{f_n\}_{n\geq 1}$ of integrable functions on $[0,1]$ such that

$$\lim_{n\to\infty} f_n(x) = 0 \qquad (1)$$

   for almost every $x \in [0,1]$.

   (a) Assuming that the functions f_n are nonnegative, find $\lim_{n\to\infty} \int_0^1 e^{-f_n(x)}\,dx$.

   (b) What can you say about this limit if the functions f_n may take both positive and negative values?

   (c) Suppose, instead of Eq. (1), you only know that $\lim_{n\to\infty} \int_0^1 |f_n(x)|\,dx = 0$.

   Do your conclusions in (a) and (b) remain valid? How?

3. (a) Define the *Fourier transform* $\hat{f}$ of an integrable function f on $\mathbb{R}^d$.
   (b) What is the Fourier transform of the Gaussian $g(x) = e^{-x(x-2)}$ (in dimension $d = 1$)?
   (c) Give an example of a function on $\mathbb{R}^d$ that lies in L^2 but not in L^1. How do you compute the Fourier transform of such a function? Please justify why your procedure works!
   (d) The Fourier transform $\mathcal{F} : f \mapsto \hat{f}$ defines a linear transformation from L^1 to L^∞.
   Is it continuous? injective? surjective?

4. Let $\mathcal{H}$ be a Hilbert space, and $A : \mathcal{H} \to \mathcal{H}$ a bounded linear operator.

   (a) Define the term *bounded* linear operator.
   Also define *weak convergence* in $\mathcal{H}$.
   (b) If $x_n \rightharpoonup x$ weakly in $\mathcal{H}$, prove that $Ax_n \rightharpoonup Ax$ weakly in $\mathcal{H}$.
   (c) If $x_n \rightharpoonup x$ weakly, and $y_n \to y$ (strongly) in $\mathcal{H}$, prove that $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

   Assume that A has the following three properties:

   - *Hermitian:* $\langle Ax, y \rangle = \langle x, Ay \rangle$ for all $x, y \in \mathcal{H}$;
   - *positive definite:* $\langle Ax, x \rangle > 0$ for all $x \neq 0$;
   - *compact:* If (x_n) is a bounded sequence, then (Ax_n) has a convergent subsequence.

   Define $\overline{\lambda} := \sup_{||x||=1} \langle Ax, x \rangle$.

   (d) Prove that the supremum is attained, i.e., there exists $\bar{x} \in \mathcal{H}$ with $||\bar{x}|| = 1$ such that $\langle A\bar{x}, \bar{x} \rangle = \overline{\lambda}$. (Consider a maximizing sequence (x_n).)

5. Formulate and prove a version of Schwarz's reflection principle for harmonic functions.

6. Suppose that $f(z)$ is meromorphic in an open subset Ω of $\mathbb{C}$, and that $K \subset \Omega$ is a compact set with oriented boundary Γ. Assume that $f(z)$ does not take the value a on Γ and has no poles on Γ. Use the residue theorem to determine what is computed by the integral

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{z^p f'(z)}{f(z) - a} dz \,,$$

where p is a positive integer.

7. (a) Show that the infinite product

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$$

represents an entire function with simple zeros at the negative integers.

(b) Define $H(z)$ by

$$\frac{1}{H(z)} = z e^{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}.$$

Prove that

$$\frac{d}{dz} \left(\frac{H'(z)}{H(z)} \right) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}.$$