DEPARTMENT OF MATHEMATICS
University of Toronto

Comprehensive Exam, 2021

Day 2: Tuesday, September 28, 4-7pm, in SS1069

Time: 3 hours. Please be brief but justify your work.
If you appeal to a standard result, be sure to carefully quote it (and verify the assumptions)

Format: 12 questions (over 2 days). Do not attempt to answer them all!
Problems come from different areas of Mathematics; work in your areas of strength, aiming for full solutions. (Passing score is 6/12)

7. Let $f, g \in L^2(S^1)$ (i.e., f and g are 2π-periodic, and square integrable over each period.) Denote the k-th Fourier coefficient of f by $\widehat{f}(k)$.

   (a) Prove that
   $$\lim_{m\to\infty} \int_{S^1} f(x)\, g(mx)\, dx = \widehat{f}(0)\, \widehat{g}(0)\,.$$

   For $m \geq 1$, write $g_{(m)}(x) := g(mx)$.

   (b) Conclude from (a) that $g_{(m)}$ converges to some limit $\bar{g}$.
   (Please specify in which sense the sequence converges, and characterize the limit $\bar{g}$.)

8. Let $A \in \mathbb{R}^{n\times m}$ be a real rectangular matrix.

   Let $\|A\|_2$ be the operator norm of A (considered a linear map from $\mathbb{R}^m$ to $\mathbb{R}^n$, with the standard Euclidean norms). Also denote by $\|A\|_F := \sqrt{\text{trace}\,(A^tA)}$ its Frobenius norm.

   (a) Show that $\|A\|_2 \leq \|A\|_F$.

   (b) Prove that for each $k \geq 1$ there exists a matrix $B \in \mathbb{R}^{n\times m}$ of rank at most k such that
   $$||A - B||_2 \leq \frac{||A||_F}{\sqrt{k}}\,.$$

   (c) There are many applications where A is a data matrix. Explain why bounds of the form you proved in (b) are useful in such applications.

9. Prove that the real projective space $\mathbb{R}P^{2n}$ for $n \geq 1$ does not admit an open cover by two orientable open subsets with connected intersection.

10. Let $S_n = X_1 + \ldots + X_n$ be a simple symmetric random walk with $S_0 = 0$ (i.e., the steps $(X_i)_{i\geq 1}$ are i.i.d. and $\mathbb{P}(X_i = \pm 1) = 1/2$.)

    Let $\tau = \min\{n \geq 5 \,:\, S_n = S_{n-5} + 5\}$.

    (a) Is τ a stopping time?

    (b) Compute $\mathbb{E}\tau$.

    Hint: If $\tau = n$ for some $n \geq 6$, what must happen in the last six steps? How can one rewrite $\mathbb{P}(\tau = n)$?

11. Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain. Consider the energy functional

$$E[u] := \int_\Omega \tfrac{1}{2}|\nabla u|^2 - \tfrac{1}{4}u^4 \, dx, \qquad u \in H_0^1(\Omega).$$

    (a) Find the Euler-Lagrange equation.

    (b) Prove that there exists a minimizer.

    (c) Is it clear that minimizers solve the Euler-Lagrange equations (in what sense)? And vice versa? (Discuss briefly; no proofs required for this part.)

12. (a) Show that there exists a 2×2 matrix A with entries in $\mathbb{R}$ such that $A^5 = I$ but $A \neq I$.

    (b) Show that there does not exist a 2×2 matrix A with entries in $\mathbb{Q}$ such that $A^5 = I$ but $A \neq I$.

    (c) Show that there exists a 4×4 matrix A with entries in $\mathbb{Q}$ such that $A^5 = I$ but $A \neq I$. Write down an explicit example of such a matrix A.