Department of Mathematics
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Topology Comprehensive Exam

September 7, 2018

Topology I

Problem 1. Let K, L, M, S be smooth manifolds.

a) Define what it means for two submanifolds of M to be transverse.
b) Let $f : K \to M$, $g : L \to M$ be smooth maps; define what it means for f, g to be transverse.
c) If $F : K \times S \to M$ is transverse to $g : L \to M$, prove that $f_s : K \to M$ is transverse to g for all s except for a set of measure zero in S, where f_s is defined by $f_s(k) = F(k, s)$.
d) Let K and L be 1-dimensional submanifolds of S^2, the unit 2-sphere. Prove that for almost all (i.e. all but a set of measure zero) rotations $R \in SO(3)$, $R(K)$ is transverse to L.

Problem 2. In standard coordinates x, y, z, the Euler vector field on $\mathbb{R}^3$ is given by $E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$.

a) Let $v = dx \wedge dy \wedge dz$. Show that $i_E(v)$ pulls back to the unit sphere $S^2 \subset \mathbb{R}^3$ to define a volume form ω.
b) Write down vector fields V_1, V_2, V_3 on $\mathbb{R}^3$ whose flows are the rotations about the x, y, z axes, respectively.
c) Show that V_1, V_2, V_3 are tangent to S^2, defining vector fields on S^2.
d) Finally, find functions f_1, f_2, f_3 on S^2 such that $i_{V_k}(\omega) = df_k, \qquad k = 1, 2, 3.$

Problem 3. Let X be a manifold and $\phi : X \to X$ a diffeomorphism. The mapping torus of (X, ϕ) is defined to be the quotient manifold $M = (X \times \mathbb{R})/\sim$, where the equivalence relation is $(x, t) \sim (\phi(x), t + 1)$.

a) If $M = U \cup V$ for open sets U, V, write down the short exact sequence of cochain complexes which relates the de Rham complexes of M, U, V, and $U \cap V$, being careful to define the maps involved.
b) Using the Mayer-Viétoris long exact sequence, compute the de Rham cohomology groups of the mapping torus of (S^n, A), where $A : S^n \to S^n$ is the antipodal map $x \mapsto -x$.

Topology II

Problem 4. Give an example of a non-trivial knot in $\mathbb{R}^3$, that is an embedding $f : S^1 \longrightarrow \mathbb{R}^3$ such that $\pi_1(\mathbb{R}^3 \setminus f(S^1))$ is not isomorphic to $\mathbb{Z}$. Prove your answer.

Problem 5.

a) Prove that each continuous map $f : \mathbb{C}P^2 \longrightarrow \mathbb{C}P^2$ has a fixed point.
b) Prove that $\mathbb{R}P^3$ is not homotopy equivalent to $\mathbb{R}P^2 \bigvee S^3$.

Problem 6. Let M be a closed n-dimensional manifold such that its fundamental group is isomorphic to the free group $\mathbb{F}_2$ with two generators. (Recall that $\mathbb{F}_2 = \mathbb{Z} * \mathbb{Z}$.)

a) Determine $H^{n-1}(M; \mathbb{Z}_2)$.
b) Prove that each two-dimensional homology class of M is spherical. (This means that for each $h \in H_2(M; \mathbb{Z})$ there exists a continuous map $f : S^2 \longrightarrow M$ such that $h = f_*([S^2])$, where $[S^2]$ denotes the fundamental homology class of the 2-dimensional sphere S^2, and f_* denotes the homomorphism $H_2(S^2; \mathbb{Z}) \longrightarrow H_2(M; \mathbb{Z})$ induced by f. Or, in other words, this means that the Hurewicz homomorphism $\pi_2(M) \longrightarrow H_2(M)$ is surjective.)