Math 344 Winter 2002

Problem Set 11

Section 8.2: 28, 30, 34

Section 8.3: 11, 18, 31, 33

Section8.5: 13, 16, 19, 20, 23, 33, 35

Section 8.6: 3, 8, 10, 13, 19, 24, 26

A. Suppose you are given interval tiles of lengths $1, 2$ and 3 in 4 different colours of 1's, 11 different colours of 2's and 6 different colours of 3's and there is an unlimited supply of each length-colour combination. How many ways are there to tile an interval of length n with these tiles? (Find a recurrence relation and solve it.)

B. Differentiate the formula

$$1 + x + x^2 + x^3 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

to obtain a formula for $\sum_{k=1}^{n} kx^k$. Differentiate again to obtain a formula for $\sum_{k=1}^{n} k^2x^k$. Now take the limit as x goes to 1 to obtain a formula for $\sum_{k=1}^{n} k^2$.

C. Find the partial fraction decomposition of

$$\frac{x^3}{(1-2x)^2(1-x)^2}$$

. This decomposition has the form

$$\frac{A}{1-2x} + \frac{B}{(1-2x)^2} + \frac{C}{1-x} + \frac{D}{(1-x)^2}.$$

D. Find a particular solution of the form $x_n = an + b$ for the recurrence relation

$$x_{n+2} - 4x_{n+1} + 4x_n = n. \tag{1}$$

Show that if $\{y_n\}$ is any solution of the relation

$$y_{n+2} - 4y_{n+1} + 4y_n = 0$$

then $x_n + y_n$ is a solution of (1). Now find the solution of (1) satisfying the initial conditions $x_0 = x_1 = 1$.

E. Use an approach as in D above to solve the recurrence relation

$$a_{n+2} - 4a_{n+1} + 4a_n = 3^n, \quad a_0 = 0, a_0 = 0. \tag{2}$$

What form should you try as a particular solution in this case?

F. (Alternative solution of problem E.) Show that the generating function $f(x)$ of the sequence $\{a_n\}$ defined by (2) in problem E satisfies

$$f(x) - 4xf(x) + 4x^2 f(x) = 1 - 3x + x^2 \sum_{i=0}^{\infty} 3^n x^n.$$

Now solve for $f(x)$ as a rational function, decompose $f(x)$ into partial fractions and expand $f(x)$ in a power series to find a formula for a_n.

G. Let a_n be the number of partitions of n in which each summand occurs at most k times and let b_n be the number of partitions of n in which no summand is a multiple of $k+1$. for example when $k = 2$

$$4 = 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 \text{ and } 4 = 4 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1,$$

so $a_4 = b_4 = 4$. Write down the generating functions for these two sequences and show that they are equal, thereby concluding that $a_n = b_n$. Hint: the generating function for $\{a_n\}$ should have factors of the form $1+y+y^2+\ldots+y^k$ where $y = x^j$. Write each of these factors as $(1 - y^{k+1})/(1 - y)$.

H. Use generating functions to show that the number of partitions of n in which no summand occurs exactly once (that is every summand occurs either 0 or two or more times) is the same as the number of partitions of n in which no summand is congruent to either 1 or 5 modulo 6.

I. Use Ferrer's diagrams to show that the number of partitions of n is the same as the number of partitions of $2n$ into exactly n summands.

J. Use Ferrer's diagrams to show that the number of partitions of n into even summands is the same as the number of partitions of n in which each summand occurs an even number of times. (This is also easy to see via generating functions.)