Math 344 Winter 2002

Problem Set 7

Section 3.5: 17, 18, 32, 33, 35, 36, 39, 40, 41, 43, 44, 45, 61, 62, 65, 73, 74, 75

A. Prove that a connected graph can be made into a strongly connected directed graph by choosing directions for the edges if and only if no edge disconnects it. Do this by providing an algorithm for directing the edges. A rough outline for such an algorithm goes as follows: Start by finding a cycle in the graph and directing its edges in a consitent way. Mark all the vertices in this cycle. If this exhausts all the vertices you are done, just direct the remaining edges arbitrarily. If unmarked vertices remain show that you can find a path from one marked vertex to another (possibly the same one) which goes through at least one unmarked vertex, direct its edges in a consistent manner and mark its vertices. The marked vertices and directed edges now form a stronly connected directed graph. If unmarked vertices remain again find a path from a marked vertex to a marked vertex. Continue in this manner...

B. Imagine an $n \times n$ chessboard and a chess piece which is allowed to move only one step forward or backward or one step to either side (a king, without its diagonal moves). For what values of n is it possible for the piece to make a trip which visits every square just once and winds up at its starting point? Prove that your answer is correct. (Hint: the colouring of a checkerboard could be helpful here.)

C. The vertices of a graph are the positions in a Pascal triangle of height n and each vertex is joined the two vertices on the same level to either side of it and also to the two vertices on the level below to either side (if there is a level below). (This graph looks like a grid of equilateral triangles.) For what values of n does this graph have a Hamiltonian circuit? Prove that your answer is correct.

D. Let $\mathcal{G}_1$ be a four cycle with an additional edge joining two of its non-adjacent vertices let $\mathcal{G}_2$ be a four cycle with an additional vertex joined to two non-adjacent vertices of the cycle. Determine the number of m-coloutings of $\mathcal{G}_1$ and $\mathcal{G}_2$. (The first one is easy. For the second one, for each pair i, j of vertices let $A_{i,j}$ be the set of m-labellings of the graph (functions from the vertices to $\{1, 2, \ldots, m\}$) such that i and j get the same label. Use inclusion exclusion to determine the cardinality of the union of the $A_{i,j}$. (It gets a bit tricky when you get to the 3-fold intersections - they will not all have the same cardinality.)

E. Recall that sink in a directed graph is a vertex with only incoming edges and a source is a vertex with only outgoing edges. Show that every acyclic directed graph has a sink and a source.

F. Prove that any tournament graph has a Hamiltonian path. (Recreate the proof you saw in class or find your own.) Prove that such a path is is unique if the tournament is acyclic. Show that there is only one acyclic tournament graph on n vertices, up to isomorphism. (What does isomorphism of directed graphs mean?)

G. Let $\mathcal{G}$ be any acyclic directed graph and think of it as representing a set of players in a tournament with an arrow from A to B if A beats B. (Not all pairs of players play.) Assign a "level" to each player as follows. All vertices with no outgoing edges (a no-win player) are at level 0. Then all vertices whose outgoing edges lead only to level 0 vertices are at level 1. In general a vertex is at level k if all its outgoing edges lead only to vertices at level $k-1$ or lower. Show that every vertex is assigned a level in this way and that these levels constitute a colouring of the graph.