Math 344 Winter 2002

Problem Set 8

The Ramsey theory problem set

This problem set is about Ramsey theory. This topic is not in the text but some notes on it are posted on the webpage. Some problems are actual applications of Ramsey's theorem, others are results in the spirit of Ramsey theory, that is recovering a moderate amount of order in a large amount of chaos. Ultimately this always comes down to the pigeonhole principle.

Don't be alarmed by the apparent length of this assignment. At least half of it's length is taken up by "hints", actually outlines of how to do the problems.

A. Recall that $R(a, b)$ denotes the least integer n such that any red-green colouring of K_n must contain either a red K_a or a blue K_b. The proof of Ramsey's two-colour theorem shows that $R(a, b) \leq R(a-1, b) + R(a, b-1)$. Explain why $R(2, a) = a$. Use this to show that $R(3, 3) \leq 6$ (in fact we know that $R(3, 3) = 6$), $R(3, 4) \leq 10$, $R(4, 4) \leq 20$, $R(3, 5) \leq 15$ and $R(5, 5) \leq 70$.

   Show that $R(3, 4) \leq 9$. Hint: Suppose the edges of K_9 are coloured red or blue and there is no red K_3 or blue K_4. Since $R(2, 4) = 4$ and $R(3, 3) = 6$ it follows that no vertex can have as many as 4 red edges or 6 blue edges adjoining it (explain) so in fact each vertex must have exactly 3 red edges and 5 blue edges adjoining it. Derive a contradiction by looking at the graph formed by the red edges alone.

   Now show that $R(3, 4) = 9$ by finding a red-blue colouring of the edges of K_8 which has no red K_3 or blue K_4. Equivalently, to avoid colouring all 28 edges, find a graph with 8 vertices which has no K_3 and also no independent set of 4 vertices. (A set of vertices in a graph is **independent** if no two are adjacent.) Hint: an 8-cycle has just two independent 4-sets.

B. Use the inequality in A above to prove that $R(a, b) \leq 2^{a+b-2}$, so $R(a, a) \leq 4^{a-1}$. Your proof should be a double induction on a and b, in the manner of the proof of Ramsey's theorem given in the notes. Don't forget to check the (infinitely many) base cases.

C Prove the three colour Ramsey theorem: given a and c there is an n such that no matter how the edges of K_n are 3-coloured there must be a monochromatic K_a. Do this by reducing to the two colour case: replace a red, blue, green colouring with a red, not-red colouring. Explain why this argument shows that $n = R(a, R(a.a))$ will do the job.

D. Let $R_c(a)$ denote the least integer n with the property that no matter how K_n is c-

coloured there must be a monochromatic K_a. The existence of $R_c(a)$ is guaranteed by Ramsey's theorem. So, $R_2(a) = R(a,a)$. Show that $R_3(a) \le 2^{a+4^a-2}$. Show that $R_{2c}(a) \le R_2(R_c(a)) \le 4^{R_c(a)}$ by an argument similar to that in C above (divide the $2c$ colours into two groups of size c). Conclude that $R_{2^n}(a) \le (\exp_4)^n(a)$ and hence $R_c(a) \le (\exp_4)^n(a)$ where $n = [\log_2(a)] + 1$. $\exp_4$ denotes the function $\exp_4(x) = 4^x$ and $\exp_4^n$ denotes the result of applying $\exp_4$ n times. To get an idea how fast $(\exp_4)^n(a)$ grows with n note that $4^{4^{4^4}}$ (supply the implicit parentheses in this expression!) has more than 200 million digits when written in decimal notation! Although better bounds are possible, all known bounds on Ramsey numbers grow very fast with c and even more so when one looks at Ramsey numbers for the colourings of r-sets (everything in this problem is about $r = 2$).

E. This problem is about $N_n^2 = \{1,\ldots,n\}^2$, the set of integer lattice points in the plane with co-ordinates between 1 and n. The **diagonal** of this set is its intersection with the line $y = x$. Given a, c show that there exists n with the following property: no matter how N_n^2 is c-coloured there exists $A \subset N_n^2$ such that the points of A^2 above the diagonal all have the same colour, the points on the diagonal all have the same colour and the points below the diagonal all have the same colour. Hint: for each 2-set $\{i,j\}$ of $\{1,\ldots,n\}$, $1 \le i < j \le n$ colour $\{i,j\}$ with the 4-tuple of colours seen on the points $(i,i), (i,j), (j,i), (j,j)$, the vertices of a square with diagonal on the diagonal of N_n^2 (so there are c^4 possible "colours") and use Ramsey's theorem to find a subset $A \subset N_n^2$ of cardinality a all of whose 2-sets have the same colour. (Of course you must use Ramsey's theorem to find the appropriate value of n in advance, before the colouring is known.) Why does A do the job?

Prove or disprove: Given a there exists n with the following property: no matter how N_n^2 is 2-coloured there exists $A \subset N_n^2$ such that all the points of A^2 have the same colour.

F. Given a, c show that there is an n with the following property: for any c-colouring of N_n^2 there are subsets A and B of N_n^2, both of cardinality a such that $A \times B$ is monochromatic.

Hint: Problem A of assignment 4 is the case $a = 2$ $c = 3$ of the present problem. This problem can be done in a similar manner: Pick k horizontal lines in the "checkerboard" and for each i, $1 \le i \le n$ colour i by the k-tuple of colours seen in the ith vertical line at the intersections with the k chosen horizontal lines (so c^k possible colours). If n is sufficiently large (how large?) by the pigeonhole principle there must be a single k-tuple of colours which occurs for a different i. Now if k is sufficiently large some colour must occur a times in this k-tuple (again pigeonhole). To make this argument precise you must first determine how large k needs to be and then how large n should be. Give an explicit value for n, in terms of a and c, which will work.

G. Say a set S of points in the plane is in **general position** if no three points in it are collinear. Given a show that there is an n with the following property: whenever S is a set of n points in the plane then there is a subset a of S with cardinality a which is either in general position or else all its points lie on one line. (Hint: colour the 3-sets of S appropriately and use Ramsey's theorem.)

H.

(i) Given any 5 points in the plane in general position show that some three of them are the vertices of an obtuse triangle ("obtuse" means one angle strictly greater than $\pi/2$).

(ii) Given a show that there is an n with the following property: for any set S of n points in the plane in general position there is a subset A of cardinality greater than a such that any three of its points form an obtuse triangle.

I. Prove the following theorem of Erdos and Szekeres: Given any sequence

$$\mathbf{a} = (a_1, a_2, \ldots, a_{mn+1})$$

of $mn + 1$ distinct real numbers $\mathbf{a}$ must have either an increasing subsequence of length $m + 1$ or a decreasing subsequence of length $n + 1$.

Hint: suppose neither kind of subsequence exists. For each i let k_i be the length of the longest increasing subsequence starting at a_i and let l_i be the length of the longest decreasing subsequence starting at a_i, so $1 \leq m_i \leq m$ and $1 \leq n_i \leq n$. Colour i with the pair (m_i, n_i). By the pigeonhole principle there must be i and j which have the same colour. Why is this a contradiction? (Suppose $i < j$ and consider the possibility of adjoining a_i to the maximal increasing or decreasing subsequence starting at a_j.)

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The last two problems are challenge problems for those students who enjoy a mathematical challenge for its own sake. I would be especially interested in seeing a solution to the last problem - I don't know how to do it!

J. Show that if every point of the plane is coloured red, green or blue then there must some two points of the same colour which are at distance one from each other.

K. Show that there is an n such that no matter how N_n^2 is 2-coloured there must be some 4 points of the same colour which form the vertices of a square whose sides are parallel to the axes.

K$'$. Here is a variant which should be easier, in principle, since you have vastly more points to work with. Show that if the points of the plane are all coloured either red or blue then there must be a square whose vertices all have the same colour. I don't insist that the sides of the square be parallel to the axes.