Math 344 Winter 2002
Problem Set 1 solutions

Disclaimer: These are intended as sketches of solutions only. There will certainly be typos and there may also be more significant errors. If you notice any significant mistakes please send me email if you are sure it is a real error, otherwise please talk to me first.

§**2.6**

8. The inductive step from 1 to 2 does not work: the two subsets in question are $\{x_1\}$ and $\{x_2\}$ which have no element in common.

29. True for $n = 1$. Now suppose $2^n + 3^n \equiv 5^n$ for some $n \geq 1$ (all congruences are to be understood modulo 6). Multiplying by 5 we get

$$5^{n+1} \equiv (2+3)(2^n + 3^n) \equiv 2^{n+1} + 3^{n+1} + 2 \cdot 3^n + 3 \cdot 2^n \equiv 2^{n+1} + 3^{n+1},$$

as the last two terms are divisible by 6 since $n \geq 1$.

§**2.7**

35. $(4n)!$ is divisible by

$$2 \cdot 4 \cdot (4+2) \cdot (2 \cdot 4) \cdot (2 \cdot 4+2) \ldots (4n) = 2^n 1 \cdot 3 \cdot 4 \cdot (2n-1) \cdot 4^n n! = 8^n 1 \cdot 3 \cdot 4 \cdot (2n-1) \cdot n!.$$

(Induction doesn't seem to be particularly helpful here.)

38. An inductive proof is straightforward. You can also do this by grouping the terms in pairs, factoring and summing an arithmetic progression, which is preferable since you don't have to know the formula in advance.

44. One line divides the plane into 2 regions. Adding another line creates 2 new regions so $2 + 2$ regions. In general if you have n lines and add one more line l then l will intersect each of the n lines in the order $l_1, l_2, \ldots l_n$, say. This means that l passes through $n + 1$ already present regions (one before l_1 one between l_1 and l_2 and so on) thereby creating $n + 1$ new regions. Thus n lines divide the plane into $2 + 2 + 3 + \ldots + n = 1 + n(n + 1)/2$ regions. This argument is inductive in spirit, if not formally. A more formal argument could incorporate an inductive proof that $2 + 2 + 3 + \ldots n = 1 + n(n + 1)/2$, or could simply appeal to the sum of an arithmetic progression, as we have done.

Chapter 2 supplement

77. $8 = 3 + 5$, $9 = 3 \cdot 3$, $10 = 2 \cdot 5$. Now for $n \geq 11$ $n - 3 \geq 8$ so by induction $n - 3 = 3k + 5l$ so $n = 3(k + 1) + 5l$. In fact we have shown that you never need more than two 5-cent stamps.

§7.1

25 $(1 + 1)^n = \sum_{i=0}^{n} C(n, i)$ by the binomial theorem. Alternately 2^n is the number of binary sequences of length n and $C(n, i)$ is the number of such sequences which contain exactly i ones.

§2.1

26. This is true if and only if $A = B$. The "if" is obvious. For the other direction suppose $A - B = B - A$ and $A \neq B$. This mean that there is an $x \in A$ which is not in B, or vice-versa. In the first case $x \in A - B$ but $x \notin B - A$, a contradiction. The other case is the same.

27. True if and only if $B \subset A$.

37. To show:

$$(A - B) \cap (A - C) = A - (B \cup C).$$

Using $\wedge$ to denote "and", x belongs to the left hand side iff

$$x \in A - B \ \wedge \ x \in A - C \iff x \in A \ \wedge \ x \notin B \ \wedge \ x \notin C \iff x \in A \ \wedge \ x \notin B \cup C$$

iff x belongs to the right hand side. This can also be shown by a Venn diagram.

A. In the expansion of $(1 + x)^{n+k}$ the coefficient of x^k is $C(n + k, k)$. Now

$$(1 + x)^{n+k} = (1 + x)^n (1 + x)^k = (\sum_{i=0}^{n} C(n, i) x^i) \ (\sum_{j=0}^{k} C(n, j) x^j)$$

Expanding this product we obtain all terms in x^k by multiplying $C(n, i)x^i$ in the first sum with $C(n, j)x^j$ in the second sum, where $j = k - i$. Adding all such terms we get $(\sum_{i=0}^{k} C(n, i)C(n, k - i)$ as the coefficient of x^k.
For the combinatorial argument just note that to choose k elements from the blue and red set we must first choose i red ones and then $k - i$ blue ones in and the number of ways to make both these choices is $C(k, i)C(n, k - i)$. Doing this for each i and adding we obtain the total number ways of choosing k elements from the blue and red set, which is $C(n + k, k)$.