Math 344 Winter 2002
Problem Set 11 solutions

Disclaimer: These are intended as sketches of solutions only. There will certainly be typos and there may also be more significant errors. If you notice any significant mistakes please send me email if you are sure it is a real error, otherwise please talk to me first.

Section 8.2

28. The recurrence relation is $r_{n+2} = r_{n+1} + 2r_n$. This can be rewritten as

$$r_{n+2} - 2r_{n+1} = 2r_n - r_{n+1} = -(r_{n+1} - 2r_n).$$

Thus we get

$$r_{n+1} - 2r_n = (-1)^n(r_1 - 2r_0) = (-1)^{n+1} \qquad \forall n \geq 1,$$

so

$$\begin{aligned} r_2 &= 2r_1 + 1 = 2 + 1, \\ r_3 &= 2r_2 - 1 = 2^2 + 2 - 1 \\ r_4 &= 2^3 + 2^2 - 2 + 1 \\ r_5 &= 2^4 + 2^3 - 2^2 + 2 - 1 \\ r_6 &= 2^5 + 2^4 - 2^3 + 2^2 - 2 + 1 \\ &\vdots \\ r_n &= 2^{n-1} + (-1)^n \sum_{i=0}^{n-2} (-2)^i = 2^{n-1} + (-1)^n \frac{1 - (-2)^{n-1}}{1+2} \\ &= \frac{2^{n+1} + (-1)^n}{3} \end{aligned}$$

Of course this can also be solved in the standard way as a second order linear recurrence relation.

30. To create a good sequence of length $n+1$ we may either write a sequence with an even number of 1's, of which there are s_n, and follow it by 0 or -1, or we may write a sequence with an odd number of 1's, of which there are $3^n - s_n$, and follow it by a 1. Thus $s_{n+1} = 2s_n + 3^n - s_n = s_n + 3^n$. Since $s_0 = 1$ it follows that $s_n = \sum_{i=0}^{n-1} 3^i = (1 + 3^n)/2$.

34. Clearly $s_1 = 2$. To move $n+1$ disks from peg one to peg three first move the top n to peg three in s_n moves then move the largest disk to peg three (1 move)

then move the other n disks (s_n moves) back to peg one then move the largest to peg three (1 move) and then the other n to peg three (s_n moves), for a total of $s_{n+1} = 3s_n + 2$ moves. Iterating this recurrence relation we find that

$$s_n = \sum_{i=0}^{n-1} 2 \cdot 3^n = 3^n - 1.$$

Section 8.3

18. Here the characteristic equation is $r^2 - 6r + 9 = 0$ so the general solution is $s_n = a3^n + bn3^n$ and the initial conditions give $a = 1$ and $b = 2$.

31. To form a sequence with sum $n + 2$ we may either append a 2 to a sequence with a sum of n or a 1 to a sequence with a sum of $n + 1$, so $s_{n+2} = s_{n+1} + s_n$, which is the Fibonacci relation. Clearly $s_1 = 1$ and $s_2 = 2$, so s_n is precisely the nth Fibonacci number, the formula for which has been worked out elsewhere.

33. You should ignore the word "distinguishable" in the statement of the problem, which suggests that colourings which are a rotation of each other should be regarded as indistinguishable. That is a more difficult problem and is not what the author intended. I apologize for not catching it earlier. What we are looking for, then, is the number of functions from the vertex set to $\{r, y, b, g\}$ which satisfy the colouring condition: no two adjacent vertices have the same colour.

    Consider a colouring of a cycle with $n + 2$ vertices, fix one vertex V and consider how the remaining vertices are coloured. If the two vertices adjacent to V have different colours then cutting V out of the necklace and tying the ends back together produces a colouring of an $n + 1$ cycle and then there are just 2 possible colours available for V so there are $2c_{n+1}$ colourings of this type of the $n+2$-cycle . If the two vertices adjacent to V have the same colour then cutting V out and melding the 2 vertices adjacent to V produces a colouring of an n-cycle and there are three possible colours for V so the number of the number of colourings of the $n+2$-cycle of this type is $3c_n$. All told we have $c_{n+2} = 2c_{n+1} + 3c_n$ for $n \geq 2$ and $c_2 = 12$, $c_3 = 24$. (Review the reasoning carefully to understand why the recurrence is not valid for $n = 1$.) Solving the recurrence relation yields $c_n = 3^n + 3(-1)^n$.

Section 8.5

16. $(1 + x + x^2 + x^3)(1 + x + x^2 + x^3 + x^4)(1 + x^2 + x^4 + x^6)$
19. $(1 + x + x^2 + x^3)(1 + x + x^2 + \ldots)$
20. $(1 + x^3 + x^6 + \ldots)(1 + x^2 + x^4 + \ldots)$, assuming an unlimited supply of each.
33. The desired generating function is the square of the function $\sum_{p \in P} x^p$, where P denotes the set of primes.

35. The desired function is the fourth power of $\sum_{i=0}^{\infty} x^{i^2}$.

Section 8.6

3. The given function is $(1-2x)^{-1}$ so its inverse is $1-2x$.

8. Setting

$$(1+x+x^3)(a+bx+cx^2+\ldots)=1,$$

and solving for $a, b, c, \ldots$ successively we get the first few terms

$$(1+x+x^3)^{-1} = 1 - x + x^2 - 2x^3 + 3x^4 - 4x^5 + 6x^6 - 9x^7 + 13x^8 + \ldots$$

There is no discernible pattern. (As a matter of interest, it is possible to factor $1+x+x^3$ into linear factors over the complex numbers, using a messy formula for the solution of a cubic, and then do a partial fraction decomposition of $(1+x+x^3)^{-1}$ to find a very messy formula involving complex numbers for the coefficients in the above series expansion.)

10.

$$(\frac{1}{3}+x^4)^{-1} = 3(1+3x^4)^{-1} = 3(1+3x^4+9x^8+27x^{12}+\ldots)$$

26. $a=1, b=2$.

A. Let t_n be the number of tilings of length n. If a tiling of length $n+3$ ends with a 1 then it is preceded by one of s_{n+2} tilings and there are 4 possibilities for the 1, so $4s_{n+2}$ tilings ending with a 1. Similarly there are $11s_{n+1}$ tilings ending with a 2 and $6s_n$ tilings ending with a 3, for a total of

$$s_{n+3} = 4s_{n+2} + 11s_{n+1} + 6s_n$$

tilings of length $n+3$. The characteristic equation of this relation is

$$r^3 - 4r^2 - 11r - 6 = 0$$

which has roots $r = -1, = -1, 6$. Thus the general solution of this recurrence is $s_n = a(-1)^n + bn(-1)^n + c6^n$. The initial conditions are $s_0 = 1, s_1 = 4, s_2 = 27$. (Explain why this value for s_0 together with the recurrence gives the right value for s_3.) Thus we have

$$a+c=1, -a-b+6c=4, a+2b+36c=27,$$

which gives $a = 13/49, b = 1/7, c = 36/49$.

B. Differentiating and simplifying we get

$$1 + 2x + 3x^2 + \ldots + nx^{n-1} = \frac{nx^{n+1} - nx^n - x^n + 1}{(x-1)^2},$$

so

$$x + 2x^2 + 3x^3 + \ldots nx^n = \frac{nx^{n+2} - nx^{n+1} - x^{n+1} + x}{(x-1)^2}$$

Differentiating again

$$\sum_{i=0}^{n-1}(i+1)^2x^i = \frac{(n^2+2n+1)x^n + (1-2n-2n^2)x^{n+1} + n^2x^{n+2} - x - 1}{(x-1)^3},$$

and taking the limit as $x \to 1$ (use L'Hopital's rule three times) we get

$$\sum_{i=1}^{n} i^2 = \frac{2n^3 + 3n^2 + n}{6}.$$

D. Plugging $x_n = an + b$ into (1) yields $x_n = n + 2$ as a particular solution of (1). (1) can be written as $T\{x_n\} = \{n\}$ where T is the linear operator $T = S^2 - 4S + 4$. If $T\{y_n\} = \{0\}$ then

$$T\{y_n + x_n\} = T\{y_n\} + T\{x_n\} = \{0 + n\} = \{n\},$$

that is $\{x_n + y_n\}$ is a solution of (1). Now $y_n = a2^n + bn2^n$ so $a2^n + bn2_n + n + 2$ is a solution of (1). Applying the initial conditions gives $a + 2 = 1, 2a + 2b + 3 = 1$ so $a = 1, b = 0$ and the solution to the initial value problem is $2^n + n + 2$.

E. The initial conditions here should have read $a_0 = 0, a_1 = 1$. I will change them to $a_0 = a_1 = 0$ as it makes the answer a little more interesting. This time we try a solution of the form $b3^n$ and find that $b = 1$. Thus $c2^n + dn2^n + 3^n$ is a solution of (2) and applying the initial conditions we find that $c = 0, d = -1$, so the solution is $a_n = --n2^n + 3^n$.

F. We have

$$\begin{aligned} f(x) =& a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots \\ -4xf(x) =& -4a_0x - 4a_1x^2 - 4a_2x^3 + \ldots \\ +4x^2f(x) =& \qquad\qquad + 4a_0x^2 + 4a_1x^3 + \ldots \end{aligned}$$

and adding we get

$$\begin{aligned} f(x)(1 - 4x + 4x^2) =& a_0 + (a_1 - 4a_0)x + 3^0x^2 + 3^1x^3 + 3^2x^4 + \ldots \\ =& 1 - 3x + x^2(1-3x)^{-1}. \end{aligned}$$

Solving for $f(x)$ we have

$$\begin{aligned}
f(x) =& \frac{1-3x}{(1-2x)^2} + \frac{x^2}{(1-2x)^2(1-3x)} \\
=& \frac{-1}{(1-2x)^2} + \frac{1}{1-2x} + \frac{1}{1-3x} \\
=& -\sum_{n=0}^{\infty}(n+1)2^n x^n + \sum_{n=0}^{\infty} 2^n x^n + \sum_{n=0}^{\infty} 3^n x^n.
\end{aligned}$$

The coefficient of x^n in the sum of these three series is $a_n = -n2^n + 3^n$.

G. The generating function for $\{a_n\}$ is

$$\begin{aligned}
a(x) =& (1+x+x^2+\ldots x^k)(1+x^2+\ldots x^{2k})(1+x^3+\ldots x^{3k})\ldots \\
=& \frac{1-x^{k+1}}{1-x}\,\frac{1-x^{2(k+1)}}{1-x^2}\,\frac{1-x^{3(k+1)}}{1-x^3}\ldots
\end{aligned}$$

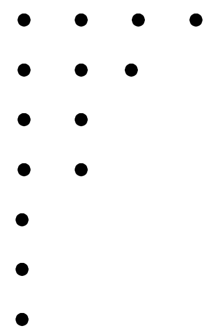
The generating function for $\{b_n\}$ is the product of factors $(1-x^n)^{-1}$ as n runs form 1 to ∞, omitting those factors where n is a multiple of $k+1$. This is evidently the same as $a(x)$ since the denominator of $a(x)$ contains **all** factors $(1-x^n)$ while the numerator contains those where n is a multiple of $k+1$, which cancel those same factors in the denominator.

H. The generating function for the first kind of partition numbers is

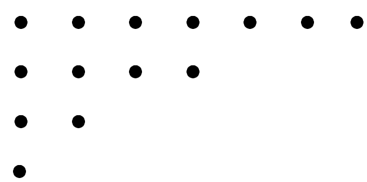
$$\begin{aligned}
a(x) =& (1+x^2+x^3+\ldots)(1+x^4+x^6+\ldots)(1+x^6+x^9+\ldots)\ldots \\
=& (\frac{1}{1-x}-x)(\frac{1}{1-x^2}-x^2)(\frac{1}{1-x^3-x^3})\ldots \\
=& \frac{1-x+x^2}{1-x}\,\frac{1-x^2+x^4}{1-x^2}\,\frac{1-x^3+x^6}{1-x^3}\ldots \\
=& \frac{1+x^3}{(1+x)(1-x)}\,\frac{1+x^6}{(1+x^2)(1-x^2)}\,\frac{1+x^9}{(1+x^3)(1-x^3)}\ldots
\end{aligned}$$

Now the denominator contains every factor $1+x^n$ and the numerator contains all such factors where n is multiple of 3, so after cancellation $a(x)$ is the product of all factors $((1+x^n)(1-x^n))^{-1}$ with the exception of of those where n is a multiple of 3, which are replaced by $(1-x^n)^{-1}$. Multiplying out those factors where n is not a multiple of 3 we get all factors $(1-x^{2n})^{-1}$, $n \neq 3k$, that is all $(1-x^n)^{-1}$ where n is congruent to 2 or 4 modulo 6. In addition we have all factors $(1-x^n)^{-1}$ where n is a multiple of 3, that is n is congruent to 0 or 3 modulo 6. In other words $a(x)$ is the product of all factors $(1-x^n)^{-1}$ where n is not congruent to 1 or 5 modulo 6. This is precisely the generating function for the second kind of partition numbers mentioned in the problem.

I. Let F be the Ferrer's diagram of a partition of $2n$ into n summands, so F is an array of $2n$ dots arranged into n rows of non-increasing length. For example $14 = 4 + 3 + 2 + 1 + 1 + 1$ is represented by the diagram



The transpose F^t of this diagram is



a partition of 14 in which the largest number is 7. Conversely if is (the Ferrer's diagram of) a partition of 14 with largest summand 7 then F^t is a partition of 14 into exactly 7 summands. In general the correspondence $F \mapsto F^t$ establishes a one-to-one correspondence between partitions of $2n$ into n summands and partitions of $2n$ in which the largest summand is n. If we have any partition of n then adjoining n to it gives a partition of $2n$ with largest summand n and this establishes a one-to-one correspondence between partitions of $2n$ with largest summand n and partitions of n.