Math 344 Winter 2002
Problem Set 2

Disclaimer: These are intended as sketches of solutions only. There will certainly be typos and there may also be more significant errors. If you notice any significant mistakes please send me email if you are sure it is a real error, otherwise please talk to me first.

§**7.2**

17(d) First order the 3 women in one 6 possible ways and then insert each man in one of 2 possible places for a total of $6 \cdot 2^3$ possible seating arrangements.

23. There are 2^4 strings starting with 1001 and 2^5 starting with 010. Since these sets are disjoint the total is $16 + 32$.

27. This problem is not hard but it is ambiguous. You have to decide whether it means how many different sets of 3 people can receive appointments or do we take into account which cities they are appointed to. For the second interpretation the answer is always 3! times the first.

30. 2^5 start with 010, 2^6 end with 11 and 2^3 do both, so the desired number is $32 + 64 - 8$.

§**7.3**

28. Answer$= P(12, 8)$.

31. In choosing 2 elements from a set with n black and n red elements (all $2n$ elements distinguishable) we may choose either all black, all red or one of each.

33. Both sides count the number of pairs A, B of subsets of a set with n elements such that $|A| = n, |B| = m$ and $A \supset B$.

§**7.4**

23. Insert 5 ones in 7 possible spots in the string 01010101010, in one of $C(5+7-1, 7-1)$ ways.

26. If the number starts with 7 there are $\frac{7!}{1!2!1!3!}$ ways to arrange the remaining digits and if it starts with a 9 there are $\frac{7!}{1!2!2!2!}$ ways. Add for the total desired number.

32. We must choose 12 coins with the restriction of no more than 10 dimes. Without restriction the number of choices is $C(14, 2)$. The number of choices with 11 or 12 dimes is 3 (explain), so the desired number is $C(14, 2) - 3$.

34. Put one ball in each urn then distribute the remaining $m - n$ balls in $C(m - n + n - 1, n - 1) = C(m - 1, n - 1)$ ways.

Chapt. 7 supplementary

50. Let

$$y_1 = 8 - x_1,\ y_2 = 6 - x_2,\ y_3 = 12 - x_3,\ y_4 = 9 - x_4.$$

Then the (y_1, y_2, y_3, y_4) are in one-to-one correspondence with the (x_1, x_2, x_3, x_4), $y_i \geq 0$ and $\sum y_i = 7$. The y_i are subject to the same upper bounds as the x_i but but all these bounds, with the exception of y_2 are guaranteed by the equation $\sum y_i = 7$. So we must count solutions of this equation such that all $y_i > 0$ and $y_2 \leq 6$. Without this restriction the number of solutions is $C(7+3, 3)$ and there just one solution with $y_2 = 7$ so the desired number is $C(10, 3) - 1$.

52. Order the n men in $n!$ ways then insert the first woman in one of $n+1-2$ possible slots the next in one of n slots, and so on, for a total of $n!(n-1)n(n+1)\ldots 2n-2 = n(n-1)(2n-2)!$ orderings.

A. For the combinatorial proof consider a set A with $n+1$ elements, colour one of them red and the remainder black. There are $C(n-1, k-1)$ subsets of A which contain the red element and $C(n-1, k)$ which don't.

C. For each odd integer k between 1 and $2n$ let

$$C_k = \{i : 1 \leq i \leq 2n, i = k \cdot 2^j \text{ for some } j \geq 0\}.$$

Since every integer can be written as an odd number times a power of 2 we have that the set of integers between 1 and $2n$ is the union of the sets C_k as k runs over the odd integers between 1 and $2n$. Since there are exactly n such sets and A has $n+1$ elements the pigeon hole principle implies that there are distinct $x, y \in A$ such that x and y belong to the same set C_k, for some k. Then, if $y > x$ we have $x = k \cdot 2^j$ and $y = k \cdot 2^{j+i}$ so x divides y. (Note that some of the C_k have just one element. Does this mean there is something wrong with the argument?)
This is a good example of a problem which is tricky because what you should prove is much more than what is asked: we get not only $x|y$ but in fact the quotient is a power of 2.