Math 344 Winter 2002
Problem Set 4 solutions

Disclaimer: These are intended as sketches of solutions only. There will certainly be typos and there may also be more significant errors. If you notice any significant mistakes please send me email if you are sure it is a real error, otherwise please talk to me first. Any errors found will be corrected.

Section 7.6

9. The sample space is the set of all permutations of the 8 husbands. We choose 4 wives to be matched with their husbands in one of $C(8,4)$ ways and then assign the remaining men to the remaining women so that no spouses are paired. The number of ways to do this is the number of **derangements** of $\{1,2,3,4\}$ that is the number of permutations which move each number to a different place. By inclusion-exclusion the number of permutations which fix at least one number is

$$C(4,1)3! - C(4,2)2! + C(4,3)1! - C(4,4)0! = 15,$$

so the number of derangements is $4! - 15 = 9$. (See p. 401 for a more detailed explanation.) Answer=$C(8,4)9/8!$.

13. First we look at colourings of V_1, V_2, V_3 and then see how to complete them. Once we have a colouring of V_1, V_2, V_3 the number of ways to colour V_4 depends on whether or not V_2 and V_3 are coloured the same so we consider the two possibilities separately. If we want V_2 and V_3 to have different colours then all of V_1, V_2, V_3 get different colours so the number of colourings is $k(k-1)(k-2)$ and then there are $k-2$ choices for V_4 so $k(k-1)(k-2)^2$ colourings of the whole graph. If V_2 and V_3 are to have the same colour then there are $k(k-1)$ colourings of V_1, V_2, V_3 and then $k-1$ ways to colour V_4 so $k(k-1)^2$ colourings. Answer=$k(k-1)(k-2)^2+k(k-1)^2$. This can also be done by inclusion-exclusion.

15. Any number between 1 and 100 which is not square free is divisible by either 2^2 or 3^2. Answer=100-25-11+2.

19. Number of seatings with:

couple number one adjacent= $2 \cdot 7!$
couples one and two adjacent=$2^2 \cdot 6!$
couples one, two and three adjacent=$2^3 \cdot 5!$
all four couples adjacent=$2^4 \cdot 4!$

So, the number of seatings with at least one couple adjacent is

$$N = 4 \cdot 2 \cdot 7! - 6 \cdot 2^2 \cdot 6! + 4 \cdot 2^3 5! - 2^4 \cdot 4!$$

and our answer is $8! - N$.

24. Here the sample space is just the permutations of $\{1, 2, 3, 4, 5\}$ and the event consists of all derangements. The number N of derangements can be computed as in # 9 above and our answer is $N/5!$.

28. $D_{n+1} - (n+1)D_n = (-1)^{n+1}$. Since $D_2 = 1$ we get $D_3 = 1,\ D_4 = 9,\ D_5 = 44,\ D_6 = 265$, etc.

29. There are $n(n-1)$ ways to distribute n distinguishable objects into 2 distinguishable urns and $n(n-1)/2$ if the urns are indistinguishable so $S(n, 2) = n(n-1)/2 - 1$. For $S(n, n-2)$ either two urns get two objects each and the rest one or one urn gets three objects so our answer is

$$C(n, 4)C(4, 2)/2 + C(n, 3).$$

30. Answer$=S(n, k)$.

    Interpretation of $m!S(n, m)$.

    Distributing n objects into m **distinguishable** urns is the same as onto functions from a set with n elements to a set with m elements. Now each distribution into m **indistinguishable** urns leads to exactly $m!$ distributions of the former kind, one for each way of labelling the urns. (Note that different labellings will lead to different distributions since the objects in the urns are distinguishable. This would be quite different if we were dealing with indistinguishable objects.) Once we have this interpretation of $m!S(m, n)$ enumerating the onto functions using inc-exc leads to the formula in # 35.

A. Select four rows as suggested and for each of the 100 columns look at the pattern of colours seen in that column. There are just 81 possible patterns so by the pigeonhole principle some pattern must occur twice, say in columns i_1 and i_2. In that pattern there must be a repeated colour, say blue for definiteness, occurring in rows j_1 and j_2, say. Then the four squares lying at the intersections of these two rows and two columns are all blue.

B. Let v stand for vowel and c for consonant. We have 5 v's and 9 c's to be arranged with no adjacent v's, which can be done in $C(10, 5)$ ways (explain). Now the v's can be replaced by the vowels in 5! ways and the c's by the consonants in $9!(3!3!2!1!)^{-1}$ ways so our answer is $C(10, 5)5!9!(3!3!2!1!)^{-1}$.