

**Math 344 Winter 2002**  
Problem Set 5 solutions

**Disclaimer:** These are intended as sketches of solutions only. There will certainly be typos and there may also be more significant errors. If you notice any significant mistakes please send me email if you are sure it is a real error, otherwise please talk to me first.

Section 3.1

24. There is no graph with 8 vertices and 29 edges since a graph with 8 vertices has at most  $C(8, 2) = 28$  edges.
37. This is not the adjacency matrix of a graph since the 1's on the diagonals would correspond to loops. It is the adjacency matrix of a multigraph.
- 42.
- (a) These are isomorphic since each is just a cycle with 5 vertices.
  - (b) These are certainly not isomorphic since the first has 7 edges and the second has 8.
  - (c) These are not isomorphic since the first has a vertex of degree 4 and the second doesn't.
43. The graphs in (a) and in (c) are isomorphic. In each case we give an informal description of a graph which completely determines the graph and applies to both graphs in questions. These descriptions are the thinking that would lead you to an isomorphism. You should still provide an explicit isomorphism in each case.
- (a) A 4-cycle with two additional vertices joined to each other, one of which is joined to two adjacent vertices of the 4-cycle and the other joined to the remaining two vertices of the 4 cycle.
  - (c) A 5-cycle with one of its vertices joined to the the two vertices to which it is not already adjacent.
- (b) These are not isomorphic as one has a vertex of degree 4 and the other does not.
46. There are just two such graphs up to isomorphism: a 4-cycle with a tail of length 1 and a 3 cycle with a tail of length 2. To see this let  $A$  be the vertex of degree 1.  $A$  is connected to a single vertex say  $B$ . If  $B$  has degree 3 then it is connected to two other vertices  $C$  and  $D$  both of degree 2 each of which must be connected to a fifth vertex  $E$  which is the first possibility. If  $B$  has degree 2 then it is connected to just to just one other vertex  $C$ . If  $C$  has degree 2 and is connected to  $D$  then  $D$  can only be connected to  $E$  (since the degrees of all previous vertices are already accounted for) and  $E$  has degree one, which is no good. Thus  $C$  must have degree

3 and is connected to  $D$  and  $E$ , which must be joined to each other, and we have the second possibility.

47. There are three such graphs: a “Y” with arms of lengths 1, 1 and 3 or 1, 2 and 2 or a disconnected graph whose components are a 2-vertex graph with a single edge and a 4 vertex graph consisting of a 3-cycle with a tail of length 1.
51. If a graph has  $k$  vertices and  $m$  edges then  $C(k, 2) \geq m$  so if we let  $k_0$  be the smallest integer such that  $C(k_0, 2) \geq m$  then  $k \geq k_0$ . On the other hand if we have  $k_0$  vertices then there  $C(k_0, 2) \geq m$  possible edges to choose so there is at least one graph with  $k_0$  vertices and  $m$  edges.
53. What we have here is a graph with vertices  $A, A', B, B', C, C', D, D'$  such that no letter is connected to its primed counterpart. Thus the degree of each vertex is at most 6 and we are told that the degrees of all vertices excluding  $A$  are different. It follows each of the degrees 0 through 6 occurs exactly once among those vertices (this is an application of the pigeonhole principle). Without going into details, one sees that  $A'$  cannot have degree 6 so say  $D$  has degree 6 and then necessarily  $D'$  has degree 0. (There is no loss of generality in assuming that it is  $D$  rather than  $D'$  that has degree 6. Now one sees that  $A'$  cannot have degree 5 so say  $C$  has degree 5 and then necessarily  $C'$  has degree 2. Continuing in this way one draws the graph exactly (up to isomorphism!) and conclude that Mrs. and Mr. Lewis each shake 3 hands.

54. Pigeonhole principle!

### Section 3.2

18. All vertices have even degree, no Euler path.
20.  $a, g, d, f, i, c, b, h, e$  is an Euler path.
21.  $c, b, d, , m, k, i, h, g, e, f, j$  is an Euler path.
22. All vertices have degree 6, no Euler path.
24.  $a, b, h, i, g, f, j, m, k, e, c, d$  is an Euler circuit.
- 30-32 The vertices are the four distinct land masses in the picture and the edges are the bridges. The vertices have degree 3, 5, 3, 3 so there is no Euler circuit. Building one new bridge leaves two vertices of odd degree so still no Euler circuit (although now there would be an Euler path). Building two new bridges (edges) with no vertex in common all vertices now have even degree so there is an Euler circuit.
40. 2, 7, 8, 9, 3, 4, 11, 10, 18, 17, 16, 6, 15, 14, 20, 19, 12, 13, 5, 1, 2 is a Hamiltonian circuit.
50. In the first graph the four edges of degree 4 form a cycle while in the second one the four edges of degree 3 do not, so the graphs are not isomorphic.
51. No, the first graph has no 3-cycle while the second does.

52. (The picture of  $K_{2,3}$  is **not** wrong, my apologies.)  $K_{m,n}$  has  $mn$  edges.
53. Since the degrees in  $K_{m,n}$  are all  $m$  on one side and  $n$  on the other this graph has an Euler circuit iff  $m$  and  $n$  are both even.
54. Since a Hamiltonian cycle must alternate between the two “sides” of  $K_{m,n}$  we see that this graph has a Hamiltonian circuit iff  $m = n$ .
- A. Since there are only two vertices via which to enter or exit the inner 4-cycle, when a Hamiltonian cycle enters the inner cycle it must visit all 4 vertices before leaving, but then it has nowhere to go, so no Hamiltonian cycle is possible. The first graph has an obvious Hamiltonian cycle.
- B. For  $\mathcal{G}_1$  take two 4-cycles glued together along a pair of consecutive edges. For  $\mathcal{G}_2$  take a 4-cycle and a 3-cycle glued together along a single edge. Draw the pictures of these informal descriptions! In  $\mathcal{G}_1$  the vertices of degree 3 are not adjacent but in  $\mathcal{G}_2$  they are, so these are not isomorphic.
- C. If  $a$  is a binary sequence we will write  $1a$  for the sequence  $a$  with a 1 appended to the left and the same for appending a 0. Suppose  $a_1, a_2, \dots, a_{2^n}$  are the sequences of a Gray code of order  $n$ . (Note that a Gray code is a sequence of sequences!) Then

$$0a_1, 0a_2, \dots, 0a_{2^n}, 1a_{2^n}, 1a_{2^n-1}, \dots, 1a_1$$

is a Gray code of order  $n + 1$ . (Note that its first and last elements differ in only there first and last entries and the same holds for the middle two elements.