Math 344 Winter 2002
Problem Set 6 solutions

Disclaimer: These are intended as sketches of solutions only. There will certainly be typos and there may also be more significant errors. If you notice any significant mistakes please send me email if you are sure it is a real error, otherwise please talk to me first.

Section 3.4

4. The chromatic number is 2 as this is a bipartite graph.

6. As there are no cycles of odd length the chromatic number is 2. Colour a vertex V_0 with the colour 0 and then use breadth first search to colour vertices 0 or 1 according to whether the distance to V_0 is even or odd.

8. Chromatic number 3.

16. See # 6 above.

28. Three colours suffice for a similar stack of any height. Colour the top ball 0 then colour $0, 1, 2, 0, 1, 2, \ldots$ periodically down the left hand boundary. Then on each row do the same periodic colouring but starting with the colour already assigned to the leftmost ball.

33. We are asked to show that the graph is K_n. If not then some vertex V_0 has degree $n-2$ or less. Colour the remaining vertices with $n-1$ colours then there is a colour available for V_0.

35. Suppose we have a blue and red edge-colouring of K_6. Fix a vertex V and colour each remaining vertex with the colour of the edge joining it to V. One colour say blue must occur for some 3 of the coloured vertices, call them V_1, V_2 and V_3. If some 2, say V_1 and V_2, of V_1, V_2 and V_3 are joined by a blue edge then V_0, V_1, V_2 form a cycle with all blue edges. Otherwise V_1, V_2, V_3 form a red cycle.

A. This is equivalent to the vertex colouring problem for the graph whose vertices are 2-element subsets of the set of n players, with an edge between two vertices whenever they have a player in common, that is their intesection is non-empty.

   For K_4 the minimal edge-colouring number is 3, as is easily seen. For K_5 it is not hard to produce a 5-colouring of the edges. In fact 5 is the minimal number because if the 10 edges are 4-coloured then there must be some 3 edges with the same colour and among any three edges there are some 2 with a vertex in common, since the whole graph has only 5 vertices. Remark: we just used the pigeon-hole principle twice, albeit in a fairly transparent way.

B. We will do the argument for 9, leaving the others as an exercise. The mumbers $1, 2, \ldots, 18$ can be divided into 9 disjoint pairs $\{1, 10\}, \{2, 11\}, \ldots \{9, 18\}$. The same goes for the next 18 numbers, the 18 after that and so on. Thus the numbers 1 to 90 are divided into 45 pairs and then we can squeeze in one more pair $\{91, 100\}$ for a total of 46 pairs and eight singleton sets left over. Considering the 46 pairs and 8 singletons as pigeonholes we have 54 pigeonholes and 55 numbers so there must be a pigeonhole which contains 2 numbers. Since that pigeonhole cannot be a singleton the result follows. Replacing 9 by 11 the set

$$\{1, \ldots, 11, 23, \ldots, 33, 45, \ldots, 55, 67, \ldots, 77, 89, \ldots, 99\}$$

has 55 elements, no two differing by 11.