Math 344 Winter 2002
Problem Set 7 solutions

Disclaimer: These are intended as sketches of solutions only. There will certainly be typos and there may also be more significant errors. If you notice any significant mistakes please send me email if you are sure it is a real error, otherwise please talk to me first.

Section 3.5

35. This graph is strongly connected as it has a cycle A, C, D, B.

36. No since the graph has a **bridge**, that is an edge whose removal disconnects it.

37. Yes since there is no bridge. You can use the algorithm in A below to find a suitable orientation of the edges.

44. This graph has a vertex of degree 3 so it doesn't even have an undirected Euler circuit.

45. Yes, since every indegree is eqaul to the corresponding outdegree. For example $a, c, g, i, j, k, h, f, e, d, b$ is a directed Euler circuit.

61. To determine the distance from S to any other vertex proceed as follows.

Step 0. Give the label 0 to S.

Step n. For each vertex V labelled $n-1$ give the label n to each unlabelled vertex at the tip of an arrow emanating from V. Stop when this step produces no new labels.

Each vertex is now labelled with its distance from S, or if it is unlabelled then it cannot be reached from S. To find a shortest path from S to T work backwards from T through vertices whose labels decrease by 1 at each step.

62. The distance is 9 and $S, A, F, L, G, B, C, H, N, T$ is a shortest path.

73. The adjacency matrix is

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

The desired numbers are the $1, 4$ and $4, 1$ entries of $A, A^2.A^3$ and A^4.

74. An isomorphism between directed graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ with vertex set $\mathcal{V}_1$ and $\mathcal{V}_2$ is a one-to-one and onto function $f : \mathcal{V}_1 \to \mathcal{V}_2$ such that (V_1, V_2) is an edge of $\mathcal{G}_1$ if and only if $(f(V_1), f(V_2))$ is an edge of $\mathcal{G}_2$.

75. The property of being stronly connected is an isomorphism invariant. This is true since. using the notation of 74 above, the image under f of a directed path in $\mathcal{G}_1$ is a directed path in $\mathcal{G}_2$.

The property of having exactly 3 sources and 2 sinks is also an isomorphism invariant.

A. Clearly the existence of a bridge (an edge whose removal disconnects the graph) makes it impossible for a graph to be made strongly connected. Now suppose no bridge exists. The following algorithm directs the edges to make a strongly connected graph.

Step 0. Find a cycle, mark its vertices and orient its edges consistently. The marked vertices and oriented edges now form a strongly connected directed graph.

General step. If unmarked vertices remain find an edge joing an marked vertex V_1 and an unmarked vertex V_2. (If this were not possible the graph would not be connected.) Then find a simple path $V_1, V_2, \ldots, V_n$ ending at a marked vertex V_n (possibly V_1 itself). Mark the vertices of this path and orient its edges in a consistent way (from V_1 to V_n or vice-versa). If before this step the marked vertices and oriented edges formed a stronly connected directed graph then clearly the same is true after this step.

Continue iterating the general step until all vertices are marked and then orient any edges which are not yet oriented in an arbitrary way.

Clearly this algorithm will do the job provided it is always possible to perform Step 0 and Step n. We will show that step 0 is possible, leaving Step n as an exercise (it is rather similar). For Step 0 choose any pair V_1, V_2 of adjacent vertices and look at the set S of vertices which can be reached from V_2 without using the edge $e = \{V_1, V_2\}$. If $V_1 \notin S$ then removing e disconnects the graph since after this removal there would be no path from V_2 to V_1. Thus there is a V_2, V_1-path which does not use e and hence there is also a simple V_2, V_1-path which does not use e. Adding V_1 at the beginning of this path gives the desired cycle.

B. This is possible precisely for **even** n. To see that it is impossible for odd n just observe that such a Hamiltonian cycle through the squares would have to alternate red and black squares so would have to have the same number of red and black squares. This forces n^2 to be even and hence n is even as well.

To construct a Hamiltonian cycle for even n start at the top left hand corner, go straight down to bottom left hand corner then straight to the bottom right hand then up one square, left to the last untraversed square, up one step , then right all the way to the right nad edge and continue zig-zagging on up this way until reaching the top right hand (coming form the one below it, since n is even) and then go across the top back to where you started.

If anyone sees a more direct way find a Hamiltonian cycle please tell me about it.

C. Denote this graph by $\mathcal{G}_n$, where n is the number of levels. I claim $\mathcal{G}_n$ always has a Hamiltonian cycle. This is clear for $n = 1, 2, 3$. Now assuming that we have constructed a Hamiltonian cycle for some $\mathcal{G}_n$ we will show how to construct one for $\mathcal{G}_{n+3}$, which completes the proof by induction. View $\mathcal{G}_{n+3}$ as $\mathcal{G}_n$ with an extra layer of vertices around its perimeter. Note that by starting the Hamiltonian cycle for $\mathcal{G}_n$ at its top vertex and traversing it in the appropriate direction we also have a Hamiltonian path for $\mathcal{G}_n$ which start at the top vertex and ends at the one below and to the left. Now start at the top vertex of $\mathcal{G}_{n+3}$ go to the vertex below and to the left, then to the top vertex of the internal $\mathcal{G}_n$, traverse the afore-mentioned Hamiltonian path for $\mathcal{G}_n$ then go one step to the left which puts us back on the left boundary of $\mathcal{G}_{n+3}$, one step below where we left it. Now continue all the way around the boundary back to the top vertex.

D. For $\mathcal{G}_2$ name the 4 edges of the cycle $1, 2, 3, 4$ and the 2 remaining edges $5, 6$. For each $i = 1, \ldots, 6$ let A_i denote the m-labellings of the vertices (that is, funtions from the vertices to $\{1, 2, \ldots, m\}$) for which the 2 vertices of edge i get the same label. We want to determine the cardinality N of the union of the A_i's, that is the m-labellings which are not colourings. A k-fold intersection of the A_i''s corresponds to a choice of k of the six edges, giving a graph with a certain number of connected components, say t. The cardinality of the intersection in question is then just m^s since each component must be labelled with a single label.

Now a little case-by-case analysis shows that for $k = 1, 2, 3$ s is always $5 - k$. For example when $k = 2$ we may have 2 edges with a vertex in common or 2 disjoint edges but in either case $s = 2$. When we come to 4-fold intersections however we find that for $A_1 \cap A_2 \cap A_3 \cap A_4$ s is 2 but for every other 4-fold intersection s is 1. Clearly every 5-fold or 6-fold intersection gives $s = 1$. Thus

$$\begin{aligned} N =& |A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6| \\ =& C(6,1)m^4 - C(6,2)m^3 + C(6,3)m^2 - (C(6,4) - 1)m \\ & - m^2 + C(6,5)m - C(6,6)m, \end{aligned}$$

and the number of m-colourings of $\mathcal{G}_2$ is $m^5 - N$.

F. Recall that a tournament graph is a directed graph with exactly one edge between any two distinct vertices. Suppose that every tournament graph with n vertices has a Hamiltonian path Let $\mathcal{G}$ be a tournament graph with $n+1$ vertices, fix any vertex V of $\mathcal{G}$ and let $\mathcal{G}'$ be the graph obtained by deleting V. $\mathcal{G}'$ has a Hamiltonian path $V_1, V_{2,\ldots,V_n}$. Label the V_i 0 or 1 according to whether the edge between V_i and V is directed towards V_i of V. If V_1 is labelled 0 then $V, V_1, V_{2,\ldots,V_n}$ is a Hamiltonian path for $\mathcal{G}$ so we may assume that V_1 is labelled 1 and similarily that V_n is labelled 0. Then ther must be some i such that V_i is labelled 1 and V_i is labelled 0, and it follows that $V_1, V_{2,\ldots,V_i,V,v_{i+1},\ldots,V_n}$ is a hamiltonian path for $\mathcal{G}$.

If a tournament graph $\mathcal{G}$ is acyclic with Hamiltonian path $V_{0,} \ldots, V_n$ then for each $i < j$ the edge between V_i and V_j must be directed towards V_j. It follows that

V_i is the unique vertex with indegree $i-1$ so the Hamiltonian path is uniquely determined. Moreover we have just seen that the Hamiltonian path dtermines the directions of all the edges uniquely so up to isomorphism there is only one acyclic tournament graph with $n+1$ vertices.