Math 344 Winter 2002
Test solutions

1. Note that $0^3 + 1^3 + 2^3$ is divisible by 9 and proceed by induction. Suppose $n^3 + (n+1)^3 + (n+2)^3$ is a multiple of 9. To show the same when n is replaced by $n+1$ it suffices to observe that

$$(n+3)^3 - n^3 = 9n^2 + 27n + 27$$

is a multiple of 9.

2.
   (a) The distribution of letters is as follows:

$$c:1,\ o:2,\ s:1,\ t:3,\ i:2,\ u:1,\ n:2,\ u:1,\ n:2,\ l:2,\ y:1,\ a:1$$

   Thus the number of arrangements is the corresponding multinomial coefficient

$$\frac{16!}{2!3!2!2!2!},$$

   where the 1!'s have all been left out.

   (b) Using 0 as a placeholder for letters other than t, first write down $t0t0t$ and then place the remaining 11 0's in the 4 available slots in $C(14,3)$ ways. The placeholders can now be filled in $13!/2!2!2!2!$ ways so the number of arrangements is $C(14,3)13!/2!2!2!2!$.

   (c) Since specifying the 5 positions where the vowels will go determines the placement of the vowels completely' this is equivalent to replacing all the vowels by a single symbol, say v and then arranging the letters with no adjacent t's. Thus as in part (b) the answer is $C(14,3)13!/2!2!6!$.

3. Let the numbers be $x_1, x_2, \ldots, x_{100}$. If any two of these are both multiples of 197 then so is the difference (or sum) and we are done so assume that at most one of them is a multiple of 197. If one of them is indeed a multiple of 197 suppose without loss of generality that it is x_1. Now look at the 199 numbers $x_1, \pm x_i, 2 \le i \le 100$. By the pigeonhole principle some two of them must fall in the same residue class modulo 197. If these two are of the form $\{x_i, x_j\}$, $\{-x_i, -x_j\}$ or $x\{x_i, -x_j\}$ with $i \ne j$ then we have $x_i - x_j$ is divisible by 197 in the first two cases and $x_i + x_j$ is divisible by 197 in the third, so we are done. The only other possibility is a pair of the form $\{x_i, -x_i\}$ with $i \ne 1$. this means $2x_i$ is divisible by 197 which implies that x_i is divisible by 197, contradicting our assumption that only x_1 could be divisible by 197. Thus this last possibility does not arise.

4. There are 10! ways to distribute the books without any restrictions. Naming the 5 women 1, 2, 3, 4 and 5 let A_i denote the set of distributions in which i gets her

own book. For $k = 1, \ldots, 5$ the cardinality of any k-fold intersection of the A_i is $(10-k)!$ (give each of the k women in the intersection her own book and then distribute the remaining $10-k$ books among the remaining $10-k$ people. Using inclusion-exclusion we have

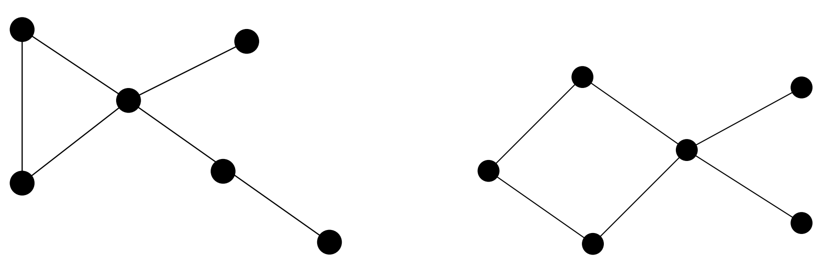
$$|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5| = 5 \cdot 9! - C(5,2)8! + C(5,3)7! - C(5,4)6! + C(5,5)5! = k$$

so the answer to the problem is $10! - k$.

5. Let C denote the Hamiltonian cycle in $\mathcal{G}'$. Since C contains exactly $n-1$ vertices and V is adjacent to at least $n/2 > (n-1)/2$ of these, by the pigeonhole principle there must be two vertices V_1 and V_2 in C which are adjacent to each other in C (that is the edge joining them lies in C) and are both adjacent to V. By starting C at V_1 we may assume that $C = V_1V_2 \ldots V_{n-1}V_1$ and then $V_1VV_2 \ldots V_{n-1}V_1$ is a Hamiltonian cycle for $\mathcal{G}$.

6. Following the hint, there must be a set S of 3 people who are either all friends to Bob, or all strangers to Bob. In the first case, if S contains no pair of friends we are done and if S does contain a pair of friends, then these two friends together with Bob form a set of 3 people who are all mutual friends. In the second case argue exactly as in the first interchanging the roles of stranger and friend. (See the solution to §3.4 #35 in Assignment 6 for the same solution expressed a little more succinctly in the language of graph theory.)

7. The two graphs are



They are not isomorphic as the first contains a 3-cycle and the other doesn't.

Now let $\mathcal{G}$ be any graph with the prescribed degree data, let A be the vertex of degree 4, let C, D, E, F be its four neighbours and let G be the other vertex. Note that $\mathcal{G}$ has 6 edges. Suppose first that two of C, D, E, F, say C and D, are adjacent. Then both C and D have degree at least 2, hence exactly 2. G must then be joined to E or F, since all other vertices already have all the edges they are supposed to have, and we are looking at the first graph above.

If no two of C, D, E, F are adjacent then at least two of them, say C and D have degree 2 hence must both be joined to G. this accounts for all the edges and we are looking at the second graph above.