ALGEBRAS OF FIBREWISE BOUNDED HOLOMORPHIC FUNCTIONS ON COVERINGS OF COMPLEX MANIFOLDS. CARTAN THEOREMS A AND B

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Abstract. We develop the elements of complex function theory within certain algebras of holomorphic functions on coverings of complex manifolds (including holomorphic extension from complex submanifolds, properties of divisors, corona type theorem, holomorphic analogue of Peter-Weyl approximation theorem, Hartogs type theorem, characterization of the uniqueness sets, etc). Our model examples are: (1) algebra of Bohr's holomorphic almost periodic functions on tube domains (i.e. the uniform limits of exponential polynomials) (2) algebra of all fibrewise bounded holomorphic functions (arising in corona problem for H^∞) (3) algebra of holomorphic functions having fibrewise limits.

Our proofs are based on the analogues of Cartan theorems A and B for coherent type sheaves on the maximal ideal spaces of these subalgebras.

Contents

1. Introduction

In the 1930-50s K. Oka and H. Cartan laid the foundations of function theory of several complex variables by introducing the notion of a coherent sheaf, and proving that:

(A) Every germ of a coherent sheaf $\mathcal{A}$ on a Stein manifold X is generated by its global sections ("Cartan theorem A").

(B) The sheaf cohomology groups $H^i(X,\mathcal{A})$ $(i \geq 1)$ are trivial ("Cartan theorem B").

Let us recall that a sheaf of modules over the sheaf of germs of holomorphic functions on X is called *coherent* if locally both this sheaf and its sheaf of relations are finitely generated. The class of coherent sheaves is closed under natural operations. Most sheaves that arise in complex analysis are coherent.

A Stein manifold is a complex manifold that admits holomorphic embedding into some $\mathbb{C}^n$.

The Cartan theorems A and B together with their numerous corollaries constitute the so-called Oka-Cartan theory of Stein manifolds. As a consequence of these theorems, one obtains existence of solutions (in algebra $\mathcal{O}(X)$ of all holomorphic functions on X) to all classical problems of function theory of several complex variables (including Cousin problems, Poincaré problem, Levi problem, the problem of extension from analytic subsets, corona problem and many others, see, e.g., [GrR]).

The further development of complex function theory was motivated, in part, by the problems that required study of behaviour of holomorphic functions satisfying some restrictions (e.g. certain growth conditions 'at infinity'). As a consequence, the questions of whether the problems of classical complex function theory can be solved within a proper subclass of $\mathcal{O}(X)$ (e.g., consisting of holomorphic L^p-functions, $1 \le p \le \infty$) started to play an important role. However, trying to incorporate restrictions on holomorphic functions such as L^p-summability, the immediate applications of the classical Oka-Cartan theory encounter considerable difficulties. In particular, one has to complement the sheaf-theoretic approach of Oka-Cartan, e.g. by integral representation formulas on complex manifolds, estimates on solutions of $\bar{\partial}$-equation, etc (cf. [HL]).

Nevertheless, the methods of Oka-Cartan theory can be extended to work within some special classes of holomorphic functions. In the present paper we obtain analogues of Cartan theorems A and B for coherent-type sheaves of the maximal ideal spaces of subalgebras $\mathcal{O}_{\mathfrak{a}}(X) \subset \mathcal{O}(X)$ of holomorphic ($\mathfrak{a}$-) functions defined on a regular covering $p : X \to X_0$ of a connected complex manifold X_0 with deck transformation group G that are

(1) bounded on subsets $p^{-1}(U_0)$, $U_0 \Subset X_0$, and

(2) for each $x \in X$ the function $G \ni g \to f(g \cdot x)$ belongs to a closed unital subalgebra $\mathfrak{a} := \mathfrak{a}(G)$ of bounded complex functions on G (with pointwise multiplication and sup-norm) invariant with respect to the action of G on $\mathfrak{a}$ by right translations: $R_g(f)(h) := f(hg)$, $f \in \mathfrak{a}$, $g, h \in G$.

Some examples of subalgebras $\mathfrak{a}$ and $\mathcal{O}_{\mathfrak{a}}(X)$ are given in Examples 1.1 and 1.3 below.

As a consequence of our Cartan type theorems A and B, we obtain within subalgebra $\mathcal{O}_{\mathfrak{a}}(X)$ the analogues of some classical results on Stein manifolds, including extension from complex submanifolds, properties of divisors, corona-type theorem, holomorphic Peter-Weyl-type approximation theorem, Hartogs-type theorem, describe uniqueness sets, etc.

In our proofs we use some results and methods of the theory of coherent-type sheaves taking values in Banach or Fréchet spaces, pioneered by Bishop and Bungart [Bu1, Bu2], and developed further by Leiterer (over finite-dimensional Stein spaces [Lt1]), Douady, Lempert (over pseudoconvex subsets of Banach spaces with unconditional bases, cf. [Lem]), and others. Similarly to [Lt1, Lem], we don't have Oka coherence lemma.

Example 1.1 (*Holomorphic almost periodic functions*)**.** The theory of almost periodic functions was created in the 1920s by H. Bohr, and shortly found numerous applications to various areas of mathematics, including number theory, harmonic analysis, differential equations (e.g. KdV equation), etc. Recall that a function $f \in \mathcal{O}(T)$ on a tube domain $T = \mathbb{R}^n + i\Omega \subset \mathbb{C}^n$, $\Omega \subset \mathbb{R}^n$ is open and convex, is called holomorphic almost periodic if the family of its translates $\{z \to f(z+s),\ z \in T\}_{s \in \mathbb{R}^n}$ is relatively compact in the topology of uniform convergence on tube subdomains $T' = \mathbb{R}^n + i\Omega'$, $\Omega' \Subset \Omega$. The cornerstone of Bohr's theory (see [Bo]) is his approximation theorem, which states that every holomorphic almost periodic function is the uniform limit (on tube subdomains T' of T) of exponential polynomials

$$z \to \sum_{k=1}^{m} c_k e^{i\langle z, \lambda_k \rangle}, \quad z \in T, \quad c_k \in \mathbb{C}, \quad \lambda_k \in \mathbb{R}^n \tag{1.1}$$

where $\langle \cdot, \lambda_k \rangle$ is the Hermitian inner product on $\mathbb{C}^n$.

The classical approach to study of holomorphic almost periodic functions exploits the fact that T is the trivial bundle with base Ω and fibre $\mathbb{R}^n$ (e.g. as in the characterization of almost periodic functions in terms of their Jessen functions defined on Ω, see, e.g. [Sh, Lev, JT, Ron, FR, To]). By considering T as a regular covering $p : T \to T_0\,(:= p(T) \subset \mathbb{C}^n)$ with the deck transformation group $\mathbb{Z}^n$,

$$p(z) := \left(e^{iz_1}, \dots, e^{iz_n}\right), \quad z = (z_1, \dots, z_n) \in T$$

(a complex strip covering an annulus if $n = 1$), we obtain

Theorem 1.2. *A function $f \in \mathcal{O}(T)$ is almost periodic if and only if $f \in \mathcal{O}_{AP}(T)$.*

Here $AP = AP(\mathbb{Z}^n)$ is the subalgebra of von Neumann's almost periodic functions on group $\mathbb{Z}^n$ (see definition in Example 3.1(2) below). This result enables us to regard holomorphic almost periodic functions on T as:

(a) holomorphic sections of a certain holomorphic Banach vector bundle on T_0;
(b) holomorphic-type functions on the fibrewise Bohr compactification of the covering $p : T \to T_0$, a topological space having some properties of a complex manifold.

As a result, we can apply the methods of multidimensional complex function theory (in particular, analytic sheaf theory and Banach-valued complex analysis) to study holomorphic almost periodic functions. In particular, even in this classical setting, we obtain new results on holomorphic almost periodic extension, Hartogs-type theorems, recovery of almost periodicity of a holomorphic function from that for its trace to a real periodic hypersurface, etc.

It is interesting to note that already in his monograph [Bo] H. Bohr uses equally often the aforementioned "trivial fibre bundle" and "regular covering" points of view on a complex strip. We note also that the Bohr compactification of a tube domain $\mathbb{R}^n + i\Omega$ in the form $b\mathbb{R}^n + i\Omega$ was used earlier in [Fav1, Fav2, Gri].

Example 1.3. (1) Let $\mathfrak{a} := \ell_\infty(G)$ be the subalgebra of all bounded functions on the deck transformation group $G \cong p^{-1}(x)$, $x \in X_0$, of covering $p : X \to X_0$.

By definition, every subalgebra $\mathcal{O}_{\mathfrak{a}}(X) \subset \mathcal{O}_{\ell_\infty}(X)$.

The subalgebra $\mathcal{O}_{\ell_\infty}(X)$ arises, e.g., in study of holomorphic L^2-functions on coverings of pseudoconvex manifolds [GHS, Br2, Br5, La], Caratheodory hyperbolicity (Liouville property) of X [LS, Lin], corona-type problems for bounded holomorphic functions on X [Br1]. Earlier, some methods similar to the ones developed in the article were elaborated for algebra $\mathcal{O}_{\ell_\infty}(X)$ in [Br1]-[Br4] in connection with corona-type problems for some subalgebras of bounded holomorphic functions on coverings of bordered Riemann surfaces, Hartogs-type theorems, integral representation of holomorphic functions of slow growth on coverings of Stein manifolds; this work was motivated by the fact that if X_0 is compact, then $\mathcal{O}_{\ell_\infty}(X) = H^\infty(X)$, the Hardy subalgebra of all bounded holomorphic functions on X (the case of special importance is when X =unit ball in $\mathbb{C}^n$).

(2) Let $\mathfrak{a} := c_0(G)$ (with card $G = \infty$) be the subalgebra of functions that admit extension to the one-point compactification of G. Then $\mathcal{O}_{c_0}(X)$ consists of holomorphic functions that have fibrewise limits at 'infinity'.

For other examples of subalgebras $\mathfrak{a}$ and $\mathcal{O}_{\mathfrak{a}}(X)$ see Sections 3.1 and 3.3.

Notation and definitions. We endow $\mathcal{O}_{\mathfrak{a}}(X)$ with the Fréchet topology of uniform convergence on subsets $p^{-1}(U_0)$, $U_0 \Subset X_0$.

By $\Lambda_c^{t,s}(X)$ we denote the space of smooth (t,s)-forms with compact supports in X, endowed with the standard topology (see, e.g. [Dem]); continuous linear functionals on $\Lambda_c^{t,s}(X)$ are called $(n-t,n-s)$-currents.

In what follows, we assume that X_0 is equipped with a path metric d_0 determined by a (smooth) hermitian metric. Let d be a semi-metric on X defined by

$$d(x_1,x_2) := d_0\big(p(x_1),p(x_2)\big), \quad x_1,x_2 \in X.$$

A function $f \in C(X)$ is called a continuous $\mathfrak{a}$-function if it is bounded and uniformly continuous with respect to semi-metric d on subsets $p^{-1}(U_0)$, $U_0 \Subset X_0$, and is such that for each $x \in X$ the function $G \ni g \to f(g \cdot x)$ belongs to $\mathfrak{a}$. We denote by $C_{\mathfrak{a}}(X)$ the subalgebra of continuous $\mathfrak{a}$-functions on X. Clearly, this subalgebra does not depend on the choice of the hermitian metric, and we have $C_{\mathfrak{a}}(X) \cap \mathcal{O}(X) = \mathcal{O}_{\mathfrak{a}}(X)$.

If $D_0 \subset X_0$ is a subdomain, we set $D := p^{-1}(D_0) \subset X$, and define $C_{\mathfrak{a}}(\bar{D})$ to be subalgebra of complex functions defined on $\bar{D}$ (the closure of D) that are bounded and uniformly continuous with respect to semi-metric d, and such that for each $x \in \bar{D}_0$ the function $G \ni g \to f(g \cdot x)$ belongs to $\mathfrak{a}$.

Over each simply connected open subset $U_0 \subset X_0$ there exists a biholomorphic trivialization $\psi : p^{-1}(U_0) \to U_0 \times G$ of covering $p : X \to X_0$, which is a morphism of fibre bundles with fibres G. We fix some system of biholomorphic trivializations, and denote for a given subset $S \subset G$

$$\Pi(U_0,S) := \psi^{-1}(U_0 \times S) \tag{1.2}$$

(see Section 4.5 for details).

2. Main results

2.1. Our approach is based on analogues of Cartan theorems A and B for coherent-type sheaves on the fibrewise compactification $c_{\mathfrak{a}}X$ of the covering $p : X \to X_0$, a fibre bunde homeomorphic to the maximal ideal space of subalgebra $\mathcal{O}_{\mathfrak{a}}(X)$ (provided that subalgebra $\mathfrak{a}$ is self-adjoint, i.e., closed with respect to complex conjugation, and X_0 is a Stein manifold, cf. Theorem 4.11), and containing the principal fibre bundle $p : X \to X_0$ as a subbundle. We describe briefly the construction of $c_{\mathfrak{a}}X$, postponing the detailed exposition till Section 4.

Let $M_{\mathfrak{a}}$ denote the maximal ideal space of algebra $\mathfrak{a}$, i.e., the space of non-zero continuous homomorphisms $\mathfrak{a} \to \mathbb{C}$ endowed with weak* topology (of $\mathfrak{a}^*$). The space $M_{\mathfrak{a}}$ is compact and Hausdorff, and every element f of $\mathfrak{a}$ determines a function $\hat{f} \in C(M_{\mathfrak{a}})$ by the formula

$$\hat{f}(\eta) := \eta(f), \quad \eta \in M_{\mathfrak{a}}.$$

Since algebra $\mathfrak{a}$ is uniform (i.e., $\|f^2\| = \|f\|^2$) and hence is semi-simple, the homomorphism $\mathfrak{a} \to C(M_{\mathfrak{a}})$ (called Gelfand transform) is an isometric embedding (see, e.g., [Gam]). We have a continuous map $j = j_{\mathfrak{a}} : G \to M_{\mathfrak{a}}$ defined by associating to each point in G its point evaluation homomorphism in $M_{\mathfrak{a}}$. This map is an injection if and only if algebra $\mathfrak{a}$ separates points of G.

Let $\hat{G}_{\mathfrak{a}}$ denote the closure of $j(G)$ in $M_{\mathfrak{a}}$ (also a compact Hausdorff space). If algebra $\mathfrak{a}$ is self-adjoint, then $\mathfrak{a} \cong C(M_{\mathfrak{a}})$ and hence $\hat{G}_{\mathfrak{a}} = M_{\mathfrak{a}}$. The action of group G on itself by right multiplication extends to the right action of G on $M_{\mathfrak{a}}$, so that $\hat{G}_{\mathfrak{a}}$ is invariant with respect to this action.

Definition 2.1. The fibrewise compactification $\bar{p} : c_{\mathfrak{a}}X \to X_0$ is defined to be the associated fibre bundle to the regular covering $p : X \to X_0$ (regarded as a principal bundle) with fibre $\hat{G}_{\mathfrak{a}}$.

(cf. Section 4.2.) Now, there exists a continuous map

$$\iota = \iota_{\mathfrak{a}} : X \to c_{\mathfrak{a}}X \tag{2.3}$$

induced by the equivariant map j. Clearly, $\iota(X)$ is dense in $c_{\mathfrak{a}}X$. If $\mathfrak{a}$ separates points of G, then ι is an injection.

Definition 2.2. A function $f \in C(c_{\mathfrak{a}}X)$ is called *holomorphic* if its pullback $\iota^* f$ is holomorphic on X. The algebra of functions holomorphic on $c_{\mathfrak{a}}X$ is denoted by $\mathcal{O}(c_{\mathfrak{a}}X)$.

For a subalgebra $\mathfrak{a}$ that is not self-adjoint we have $\mathcal{O}_{\mathfrak{a}}(X) \hookrightarrow \mathcal{O}(c_{\mathfrak{a}}X)$ (see Proposition 4.6 below). This embedding is an isomorphism if $\mathfrak{a}$ is self-adjoint; in this case, we can work with algebra $\mathcal{O}(c_{\mathfrak{a}}X)$ instead of subalgebra $\mathcal{O}_{\mathfrak{a}}(X)$.

We define holomorphic functions on open subsets of $c_{\mathfrak{a}}X$ analogously, and therefore obtain the structure sheaf $\mathcal{O} := \mathcal{O}_{c_{\mathfrak{a}}X}$ of germs of holomorphic functions on $c_{\mathfrak{a}}X$. Now, a *coherent sheaf* $\mathcal{A}$ on $c_{\mathfrak{a}}X$ is a sheaf of modules over $\mathcal{O}$ such that every point in $c_{\mathfrak{a}}X$ has a neighbourhood U over which, for any $N \geq 1$, there is a free resolution of $\mathcal{A}$ of length N, i.e., an exact sequence of sheaves of modules of the form

$$\mathcal{O}^{m_N}|_U \xrightarrow{\varphi_{N-1}} \dots \xrightarrow{\varphi_2} \mathcal{O}^{m_2}|_U \xrightarrow{\varphi_1} \mathcal{O}^{m_1}|_U \xrightarrow{\varphi_0} \mathcal{A}|_U \longrightarrow 0, \tag{2.4}$$

where φ_i, $0 \leq i \leq N-1$ denote homomorphisms of sheaves of modules. If $X = X_0$ and $p = \mathrm{Id}$, then this definition gives the classical definition of a coherent sheaf on a complex manifold.

Let X_0 be a Stein manifold, $\mathcal{A}$ a coherent sheaf on $c_{\mathfrak{a}}X$.

Theorem 2.3 (Cartan A). *Each stalk ${}_x\mathcal{A}$ ($x \in c_{\mathfrak{a}}X$) is generated by sections $\Gamma(c_{\mathfrak{a}}X, \mathcal{A})$ as an ${}_x\mathcal{O}$-module.*

Theorem 2.4 (Cartan B). *The sheaf cohomology groups $H^i(c_{\mathfrak{a}}X, \mathcal{A}) = 0$, $i \geq 1$.*

The collection of open subsets of X of the form $V = \iota^{-1}(U)$, where $U \subset c_{\mathfrak{a}}X$ is open, forms a Hausdorff topology on X, denoted by $\mathcal{T}_{\mathfrak{a}}$.

Suppose that subalgebra $\mathfrak{a}$ is self-adjoint. We define holomorphic $\mathfrak{a}$-functions on $V \in \mathcal{T}_{\mathfrak{a}}$ by

$$\mathcal{O}_{\mathfrak{a}}(V) := \iota^* \mathcal{O}(U),$$

where $U \subset c_{\mathfrak{a}}X$ is open and such that $V = \iota^{-1}(U)$.

In the rest of this section we assume that subalgebra $\mathfrak{a}$ is self-adjoint unless otherwise stated.

2.2. Using Theorem 2.4, we obtain the following result on extension within subalgebra $\mathcal{O}_{\mathfrak{a}}(X)$.

Definition 2.5. An open cover $\mathcal{V}$ of X is called $\mathcal{T}_{\mathfrak{a}}$-fine if it is the pullback by ι of an open cover of $c_{\mathfrak{a}}X$, and for every $V \in \mathcal{V}$ the projection $\bar{p}(V) \Subset X_0$ (e.g. $\mathcal{V} = p^{-1}(\mathcal{V}_0)$, where $\mathcal{V}_0$ is an open cover of X_0 by relatively compact subsets).

Definition 2.6. A closed subset $Z \subset X$ is called a complex $\mathfrak{a}$-submanifold of codimension $k \leq n := \dim_{\mathbb{C}} X_0$ if there exists a $\mathcal{T}_{\mathfrak{a}}$-fine open cover $\mathcal{V}$ of X such that for each $V \in \mathcal{V}$ there are functions $h_1, \dots, h_k \in \mathcal{O}_{\mathfrak{a}}(V)$ that satisfy:

(1) $Z \cap V = \{x \in V \, : \, h_1(x) = \dots = h_k(x) = 0\}$;
(2) maximum of moduli of determinants of all $k \times k$ submatrices of the Jacobian matrix of the map $x \to \big(h_1(x), \dots, h_k(x)\big)$ (with respect to local coordinates on V pulled back from a coordinate chart on X_0 containing the closure of $\bar{p}(V)$) is uniformly bounded away from zero on $Z \cap V$.

Some examples of complex $\mathfrak{a}$-submanifolds are given in Section 3.5 below.

Theorem 2.7 (Characterization of complex $\mathfrak{a}$-submanifolds)**.** *Suppose that X_0 is a Stein manifold. Then a closed subset $Z \subset X$ is a complex $\mathfrak{a}$-submanifold of codimension $k \leq n$ if and only if there exists at most countable collection of functions $f_i \in \mathcal{O}_{\mathfrak{a}}(X)$, $i \in I$, such that*

(i) $Z = \{x \in X : f_i(x) = 0 \text{ for all } i \in I\}$,

(ii) for each $x_0 \in Z$ there exists a neighbourhood $V \in \mathfrak{T}_{\mathfrak{a}}$ and functions $f_{i_1}, \dots f_{i_k}$ such that $Z \cap V = \{x \in V : f_{i_1} = \dots = f_{i_k} = 0\}$ and the maximum of moduli of determinants of all $k \times k$ submatrices of the Jacobian matrix of the map $x \to \big(f_{i_1}(x), \dots, f_{i_k}(x)\big)$ (with respect to local coordinates on V pulled back from X_0) is non-zero at x_0.

Definition 2.8. A function $f \in \mathcal{O}(Z)$ on a complex $\mathfrak{a}$-submanifold $Z \subset X$ is called a holomorphic $\mathfrak{a}$-function if it admits extension to a function in $C_{\mathfrak{a}}(X)$.

The subalgebra of holomorphic $\mathfrak{a}$-functions on Z is denoted by $\mathcal{O}_{\mathfrak{a}}(Z)$. Alternatively, subalgebra $\mathcal{O}_{\mathfrak{a}}(Z)$ can be defined in terms of currents, cf. Section 3.2.

Theorem 2.9. *Suppose that X_0 is a Stein manifold, $Z \subset X$ is a complex $\mathfrak{a}$–submanifold, and $f \in \mathcal{O}_{\mathfrak{a}}(Z)$. Then there is $F \in \mathcal{O}_{\mathfrak{a}}(X)$ such that $F|_Z = f$.*

Example 2.10. Suppose that Z_1, $Z_2 \subset T := \mathbb{R}^n + i\Omega \subset \mathbb{C}^n$ (where $\Omega \subset \mathbb{R}^n$ is convex) are non-intersecting smooth complex hypersurfaces that are periodic with respect to the usual action of $\mathbb{R}^n$ on T by translations, possibly with different periods, and $f_1 \in \mathcal{O}(Z_1)$, $f_2 \in \mathcal{O}(Z_2)$ are holomorphic periodic with respect to these periods functions. By Theorem 2.9 there is a holomorphic almost periodic function $F \in \mathcal{O}_{AP}(T)$ such that $F|_{Z_i} = f_i$, $i = 1, 2$.

It is not difficult to construct an example of a subset Z satisfying (1) but not (2) in Definition 2.6, such that $\mathcal{O}_{\mathfrak{a}}(X)|_Z \subsetneq C_{\mathfrak{a}}(X) \cap \mathcal{O}(Z)$.

In some cases the requirement that subalgebra $\mathfrak{a}$ is self-adjoint is not necessary for existence of an extension. The following example gives an alternative approach to study of $\mathcal{O}_{\mathfrak{a}}(X)$. Namely, we have an equivalent presentation of functions in $\mathcal{O}_{\mathfrak{a}}(X)$ as holomorphic sections of a holomorphic Banach vector bundle $\tilde{p} : C_{\mathfrak{a}}X_0 \to X_0$ associated to the principal fibre bundle $p : X \to X_0$ and having fibre $\mathfrak{a}$, defined as follows. The regular covering $p : X \to X_0$ is a principal fibre bundle with structure group G, so there exists an open cover $\{U_{0,\gamma}\}$ of X_0 and a locally constant cocycle $\{c_{\delta\gamma} : U_{0,\gamma} \cap U_{0,\delta} \to G\}$, so that the covering $p : X \to X_0$ can be obtained from the disjoint union $\sqcup_\gamma U_{0,\gamma} \times G$ by the identification

$$U_{0,\delta} \times G \ni (x, g) \sim (x, g\, c_{\delta\gamma}(x)) \in U_{0,\gamma} \times G \quad \text{for all } x \in U_\gamma \cap U_{0,\delta}, \tag{2.5}$$

where projection p is induced by the projections $U_{0,\gamma} \times G \to U_{0,\gamma}$ (see, e.g., [Hz]). Then $C_{\mathfrak{a}}X_0$ is a fibre bundle associated to $p : X \to X_0$ and having fibre $\mathfrak{a}$. The fibre bundle $C_{\mathfrak{a}}X_0$ is obtained from the disjoint union $\sqcup_\gamma U_{0,\gamma} \times \mathfrak{a}$ by the identification

$$U_\delta \times \mathfrak{a} \ni (x, f) \sim (x, R_{c_{\delta\gamma}(x)}(f)) \in U_\gamma \times \mathfrak{a} \quad \text{for all } x \in U_\gamma \cap U_\delta. \tag{2.6}$$

The projection $\tilde{p}$ is induced by projections $U_\gamma \times \mathfrak{a} \to U_\gamma$. This is a holomorphic Banach vector bundle. Let $\mathcal{O}(C_{\mathfrak{a}}X_0)$ be the set of (global) holomorphic sections of $C_{\mathfrak{a}}X_0$. This is a Fréchet algebra with respect to the usual pointwise operations and the topology of uniform convergence on compact subsets of X_0.

Proposition 2.11. $\mathcal{O}_{\mathfrak{a}}(X) \cong \mathcal{O}(C_{\mathfrak{a}}X_0)$.

As a consequence of Proposition 2.11, we obtain the following result of extension within the class of holomorphic $\mathfrak{a}$-functions.

Proposition 2.12. *Let M_0 be a closed complex submanifold of a Stein manifold X_0, $M := p^{-1}(M_0)$, $D_0 \subset X_0$ is Levi strictly pseudoconvex (see, e.g., [GR]), $D := p^{-1}(D_0)$, and $f \in \mathcal{O}_{\mathfrak{a}}(M \cap D)$ is bounded. Then there exists a bounded function $F \in \mathcal{O}_{\mathfrak{a}}(D)$ such that $F|_{M\cap D} = f|_{M\cap D}$.*

Indeed, the subalgebra $\mathcal{O}_{\mathfrak{a}}(M)$ is isomorphic to the algebra $\mathcal{O}(C_{\mathfrak{a}}X)|_{M_0}$ of holomorphic sections of bundle CX over M_0. Since X_0 is Stein, there exist holomorphic Banach vector bundles $p_1 : E_1 \to X_0$ and $p_2 : E_2 \to X_0$ with fibres B_1 and B_2, respectively, such that $E_2 = E_1 \oplus C_{\mathfrak{a}}X_0$ (the Whitney sum) and E_2 is holomorphically trivial, i.e., $E_2 \cong X_0 \times B_2$ (see, e.g. [ZK]). Thus, any holomorphic section of E_2 can be naturally identified with a B_2-valued holomorphic function on X_0. By $q : E_2 \to C_{\mathfrak{a}}X_0$ and $\iota : C_{\mathfrak{a}}X_0 \to E_2$ we denote the corresponding quotient and embedding homomorphisms of the bundles so that $q \circ \iota = \mathrm{Id}$. (Similar identifications hold for bundle CD.) For a given function $f \in \mathcal{O}(C_{\mathfrak{a}}X_0)|_{M_0}$ consider its image $\tilde{f} := \iota(f)$, a B_2-valued holomorphic function on M_0, and apply to it the integral representation formula from [HL] asserting the existence of a bounded function $\tilde{F} \in \mathcal{O}(D_0, B_2)$ such that $\tilde{F}|_{M_0\cap D_0} = \tilde{f}|_{M_0\cap D_0}$. Finally, we define $F := q(\tilde{F})$.

In fact, this method allows to obtain similar extension results for holomorphic functions on X whose restriction to each fibre belongs to some Banach space, and is possibly unbounded, see [Br4]. It is not yet clear to what extent Theorem 2.9 depends on the assumption that subalgebra $\mathfrak{a}$ is self-adjoint.

2.3. Recall that an effective (Cartier) divisor E on X is given by an open cover $\{U_\alpha\}$ of X and functions $f_\alpha \in \mathcal{O}(U_\alpha)$, f_α not identically zero on any open subset of U_α, such that

$$f_\alpha = d_{\alpha\beta} f_\beta \quad \text{on } U_\alpha \cap U_\beta \text{ for a nowhere zero function } d_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta), \tag{2.7}$$

for all α, β. The collection of effective divisors on X is denoted by $\mathrm{Div}(X)$.

Let T_E be the integration current of a divisor $E \in \mathrm{Div}(X)$, i.e.,

$$(T_E, \varphi) := \int_E \varphi, \quad \varphi \in \Lambda_c^{n-1,n-1}(X)$$

Here we use the fact that locally (in the usual topology on X) divisor E admits presentation as a collection of analytic hypersurfaces with prescribed multiplicities.

The divisors $E = \{(U_\alpha, f_\alpha\}$, $E' = \{(V_\beta, g_\beta)\}$ in $\mathrm{Div}(X)$ are said to be equivalent if there exists a refinement $\{W_\gamma\}$ of both covers $\{U_\alpha\}$ and $\{V_\beta\}$ and nowhere zero holomorphic functions c_γ on W_γ such that $f_\alpha|_{W_\gamma} = c_\gamma g_\beta|_{W_\gamma}$ for $W_\gamma \subset U_\alpha \cap V_\beta$. It is easy to see that E, $E' \in \mathrm{Div}(X)$ are equivalent if and only if $T_E = T_{E'}$.

In the next definitions we assume that algebra $\mathfrak{a}$ is self-adjoint.

Definition 2.13. A divisor $E \in \mathrm{Div}(X)$ is called an (effective) $\mathfrak{a}$-divisor if in the above definition (2.7) of a divisor on X we have

(1) $\{U_\alpha\}$ is a $\mathcal{T}_{\mathfrak{a}}$-fine open cover,

(2) $f_\alpha \in \mathcal{O}_{\mathfrak{a}}(U_\alpha)$, for all α,

(3) $f_\alpha = d_{\alpha\beta} f_\beta$ on $U_\alpha \cap U_\beta \neq \varnothing$ for some $d_{\alpha\beta} \in \mathcal{O}_{\mathfrak{a}}(U_\alpha \cap U_\beta)$ whose modulus is uniformly bounded away from zero on every open subset $V \subset U_\alpha \cap U_\beta$ with the property that the closure of $\iota(V)$ is contained in an open subset $W \subset c_{\mathfrak{a}}X$ such that $U_\alpha \cap U_\beta = \iota^{-1}(W)$, for all α, β.

We note that for some algebras $\mathfrak{a}$, e.g. holomorphic almost periodic functions (cf. Example 1.1 and Section 3.3) the $\mathfrak{a}$-divisors can be defined equivalently in terms of their currents of integration, cf. Proposition 3.1.

The collection of $\mathfrak{a}$-divisors is denoted by $\mathrm{Div}_{\mathfrak{a}}(X)$.

The basic example of an $\mathfrak{a}$-divisor is the divisor E_f of a function $f \in \mathcal{O}_{\mathfrak{a}}(X)$, called an $\mathfrak{a}$-principal divisor. There are, however, $\mathfrak{a}$-divisors that are not $\mathfrak{a}$-principal, cf. Section 3.5(4). In fact, the Čech cohomology group $H^2(c_{\mathfrak{a}}T, \mathbb{Z})$, measuring the deviation of an $\mathfrak{a}$-divisor from being $\mathfrak{a}$-principal (cf. proof of Theorem 2.15), is in general non-trivial (e.g. in the setting of Example 1.1 the fibrewise compactification $c_{AP}T$ is an inverse limit of smooth principal fibre bundles with compact Abelian groups $(S^1)^m \times \oplus_{k=1}^{l} \mathbb{Z}/n_k\mathbb{Z}$ as their fibres, cf. Example 4.3 below).

This naturally leads to the following problem, first considered (in a special case) in [FRR]:

> Does there exist a class of functions $\mathfrak{C}_{\mathfrak{a}} \subset \mathcal{O}(X)$ such that for each function from $\mathfrak{C}_{\mathfrak{a}}$ its divisor is equivalent to a divisor in $\mathrm{Div}_{\mathfrak{a}}(X)$, and conversely, any divisor in $\mathrm{Div}_{\mathfrak{a}}(X)$ is equivalent to a principal divisor determined by a function in $\mathfrak{C}_{\mathfrak{a}}$?

If $X = \{z \in \mathbb{C} : a < \mathrm{Im}\ (z) < b\}$ and $\mathfrak{a} = AP(\mathbb{Z})$ (cf. Example 1.1), then by [FRR]

$$\mathfrak{C}_{AP} = \{f \in \mathcal{O}(X) : |f| \in C_{AP}(X)\}.$$

The proof in [FRR] uses certain properties of almost periodic currents (see [Fav2] for an extension of this result to several variables). Using a sheaf-theoretic approach, we obtain

Theorem 2.14. (1) *Suppose that X_0 is a non-compact Riemann surface, and X is the universal covering of X_0. Then for every divisor $E \in \mathrm{Div}_{\mathfrak{a}}(X)$ there exists a function $f \in \mathcal{O}(X)$ with $|f| \in C_{\mathfrak{a}}(X)$ such that E is equivalent to the principal divisor $E_f \in \mathrm{Div}(X)$.*

(2) *Conversely, for any complex manifold X_0, let $f \in \mathcal{O}(X)$, $|f| \in C_{\mathfrak{a}}(X)$, and suppose that there exists an open subset $Y \Subset X$ such that for any net $\{g_\alpha\} \subset G$ the translates $\{x \mapsto f(g_\alpha \cdot x)\}$ do not converge uniformly on $\bar{Y}$ to zero; then E_f is equivalent to a divisor $E \in \mathrm{Div}_{\mathfrak{a}}(X)$.*

The assertion of Theorem 2.14 can be refined for holomorphic almost periodic functions on coverings of complex manifolds, cf. Section 3.3.

If subalgebra $\mathfrak{a}$ is such that the covering dimension of $\hat{G}_{\mathfrak{a}}$ is zero (e.g. $\mathfrak{a} = \ell_\infty$ or $AP_{\mathbb{Q}}(\mathbb{Z}^n)$, cf. Examples 3.1(3) and 4.1(3), (4)), then the assertion of Theorem 2.14 follows trivially (cf. Section 3.4).

The second Cousin problem asks whether a given divisor $E \in \mathrm{Div}_{\mathfrak{a}}(X)$ is $\mathfrak{a}$-principal.

Theorem 2.15 (2nd Cousin)**.** *Let X_0 be a Stein manifold, $E \in \mathrm{Div}_{\mathfrak{a}}(X)$.*

If X_0 is homotopically equivalent to an open subset $Y_0 \subset X_0$ such that the restriction of E to $Y := p^{-1}(Y_0)$ is equivalent to an $\mathfrak{a}$-principal divisor, then E itself is equivalent to an $\mathfrak{a}$-principal divisor.

In particular, if $\mathrm{supp}(E) \cap Y = \varnothing$, then E is equivalent to an $\mathfrak{a}$-principal divisor.

Here $\mathrm{supp}(E)$ is the set of zeros of functions determining divisor E.

For the subalgebra of Bohr's holomorphic almost periodic functions (cf. Example 1.1; in the case X and Y are tube domains and $\mathfrak{a} = AP(\mathbb{Z}^n)$) this theorem is due to [FRR] ($n = 1$) and [Fav1] ($n \geq 1$). The proof in [FRR] uses Arakelyan's theorem and gives an explicit construction of the holomorphic almost periodic function that determines the principal divisor. Our proof, similarly to the proof in [Fav1], is sheaf-theoretic.

2.4. A classical result by H. Bohr states that if a holomorphic function f on a complex strip $T := \{z \in \mathbb{C} : \text{Im } (z) \in (a,b)\}$, bounded on closed substrips, is continuous almost periodic on a horizontal line $\mathbb{R} + ic$, $c \in (a,b)$, then f is holomorphic almost periodic on T.

We extend this result to a general algebra $\mathcal{O}_{\mathfrak{a}}(X)$ as follows.

Let X_0 be a Stein manifold, $U_0 \subset X_0$ be an open simply connected set, $Z_0 \subset U_0$ a uniqueness set for holomorphic functions in $\mathcal{O}(U_0)$, and $K \subset G$ be such that $\cup_i K g_i = G$ for some $g_1, \dots, g_m \in G$. Suppose that $Z \subset X$ is a set of the form

$$Z = \{x \in X : \rho_a(x) = 0,\ a \in A\},$$

where $\{\rho_a\}_{a \in A} \subset C_{\mathfrak{a}}(X)$. We assume that Z has the property that $p^{-1}(Z_0) \cap \Pi(U_0, K) \subset Z$ (cf. (1.2)). By $C_{\mathfrak{a}}(Z)$ we denote the restrictions to Z of functions in $C_{\mathfrak{a}}(X)$.

Theorem 2.16. *If $f \in \mathcal{O}_{\ell_\infty(G)}(X)$ and $f|_Z \in C_{\mathfrak{a}}(Z)$, then $f \in \mathcal{O}_{\mathfrak{a}}(X)$.*

It follows that in Bohr's result the line $\mathbb{R} + ic$ can be replaced, e.g., with a periodic curve.

As an example of set Z_0 we can take any real hypersurface in X_0 or, more generally, a set of the form $Z_0 := \{x \in X_0 : \rho_1(x) = \cdots = \rho_d(x) = 0\}$, where $\rho_1, \dots, \rho_d$ are real-valued differentiable functions on X_0, $d \le n$, and $\partial\rho_1(x_0) \wedge \cdots \wedge \partial\rho_d(x_0) \ne 0$ for some $x_0 \in Z_0$ (see, e.g., [Bog]).

Using the result in [Br3], we obtain the following Hartogs-type theorem.

Theorem 2.17. *Let $n \ge 2$, $D_0 \Subset X_0$ be a subdomain with a connected piecewise smooth boundary ∂D_0 contained in a Stein open submanifold of X_0, and $D := p^{-1}(D_0)$. Suppose that $f \in C_{\mathfrak{a}}(\partial D)$ satisfies tangential CR equations on ∂D , i.e., for any $\omega \in \Lambda_c^{n,n-2}(X)$*

$$\int_{\partial Y} f\ \bar\partial\omega = 0.$$

Then there exists a function $F \in \mathcal{O}_{\mathfrak{a}}(D) \cap C_{\mathfrak{a}}(\bar D)$ such that $F|_{\partial D} = f$.

In particular, Theorem 2.17 implies that if $n \ge 2$, then each continuous almost periodic function on the boundary $\partial T = \mathbb{R}^n + i\partial\Omega$ of a tube domain $T := \mathbb{R}^n + i\Omega \subset \mathbb{C}^n$, $\Omega \Subset \mathbb{R}^n$ is a domain with piecewise-smooth boundary $\partial\Omega$, satisfying tangential CR equations on ∂T, admits a continuous extension to a holomorphic almost periodic function in $\mathcal{O}_{AP}(T) \cap C(\bar T)$.

2.5. We extend Bohr's approximation theorem for holomorphic almost periodic functions (cf. Introduction) to an arbitrary subalgebra $\mathcal{O}_{\mathfrak{a}}(X)$ as follows. We do not assume that subalgebra $\mathfrak{a}$ is self-adjoint.

Let $\mathfrak{a}_\iota$ ($\iota \in I$) be a collection of closed subspaces of $\mathfrak{a}$ such that

(1) $\mathfrak{a}_\iota$ are invariant with respect to the action of G on $\mathfrak{a}$ by right translates (i.e., if $f \in \mathfrak{a}_\iota$, then $R_g(f) \in \mathfrak{a}_\iota$, $g \in G$, cf. Introduction),

(2) the family $\{\mathfrak{a}_\iota : \iota \in I\}$ forms a direct system ordered by inclusion, and

(3) the linear space $\mathfrak{a}_0 := \bigcup_{\iota \in I} \mathfrak{a}_\iota$ is dense in $\mathfrak{a}$.

The model examples of subspaces $\mathfrak{a}_\iota$ are given in Section 3.6 below.

Let $\mathcal{O}_\iota(X)$ be the space of holomorphic functions $f \in \mathcal{O}_{\mathfrak{a}}(X)$ such that for every $x_0 \in X_0$ function

$$g \mapsto f(g \cdot x), \quad g \in G, \quad x \in F_{x_0}$$

belongs to $\mathfrak{a}_\iota$. Let $\mathcal{O}_0(X)$ be the $\mathbb{C}$-linear hull of spaces $\mathcal{O}_\iota(X)$ with ι varying over I.

Theorem 2.18. *If X_0 is a Stein manifold, then $\mathcal{O}_0(X)$ is dense in $\mathcal{O}_{\mathfrak{a}}(X)$.*

If $\mathfrak{a} = AP(G)$ (cf. Sections 3.1(2) and 3.3), then this theorem may be viewed as a holomorphic analogue of Peter-Weyl approximation theorem (von Neumann approximation theorem).

We note that, together with the example of Section 3.7(1), this theorem gives another proof of Theorem 1.2.

3. Examples

3.1. **Examples of subalgebras $\mathfrak{a}$.** In addition to $\ell_\infty(G)$, $c_0(G)$, $AP(\mathbb{Z}^n)$, we mention the following examples of self-adjoint subalgebras of functions on G.

(1) If group G is residually finite (respectively, residually nilpotent), i.e., for any element $t \in G$, $t \neq e$, there exists a normal subgroup $G_t \not\ni t$ such that G/G_t is finite (respectively, nilpotent), we consider the closed algebra $\hat{\ell}_\infty(G) \subset \ell_\infty(G)$ generated by pullbacks to G of algebras $\ell_\infty(G/G_t)$, for all G_t as above.

(2) Recall that a (continuous) bounded function f on a (topological) group G is called almost periodic if the families of its left and right translates

$$\{t \to f(st)\}_{s\in G}, \quad \{t \to f(ts)\}_{s\in G}$$

are relatively compact in $\ell_\infty(G)$ (J. von Neumann [Ne]). (It was proved in [Ma] that the relative compactness of either the left of the right family of translates already gives the almost periodicity on G.) The algebra of almost periodic functions on G is denoted by $AP(G)$.

The basic example of almost periodic functions on G is given by the matrix elements of the finite-dimensional irreducible unitary representations of G.

Recall that group G is called *maximally almost periodic* if its finite-dimensional irreducible unitary representations separate points. Equivalently, G is maximally almost periodic iff it admits a monomorphism into a compact topological group.

Any residually finite group belongs to this class. In particular, $\mathbb{Z}^n$, finite groups, free groups, finitely generated nilpotent groups, pure braid groups, fundamental groups of three dimensional manifolds are maximally almost periodic.

We denote by $AP_0(G) \subset AP(G)$ the space of functions

$$t \to \sum_{k=1}^{m} c_k \sigma_{ij}^k(t), \quad t \in G, \quad c_k \in \mathbb{C}, \quad \sigma^k = (\sigma_{ij}^k),$$

where σ^k $(1 \leq k \leq m)$ are finite-dimensional irreducible unitary representations of G. The von Neumann's approximation theorem states that $AP_0(G)$ is dense in $AP(G)$ [Ne].

In particular, the algebra $AP(\mathbb{Z}^n)$ of almost periodic functions on $\mathbb{Z}^n$ contains as a dense subset the exponential polynomials $t \to \sum_{k=1}^m c_k e^{i\langle\lambda_k,t\rangle}$, $t \in \mathbb{Z}^n$, $\lambda_k \in \mathbb{R}^n$. Here $\langle\lambda_k,\cdot\rangle$ denotes the linear functional defined by λ_k.

(3) The algebra $AP_{\mathbb{Q}}(\mathbb{Z}^n)$ of almost periodic functions on $\mathbb{Z}^n$ with rational spectra. This is the subalgebra of $AP(\mathbb{Z}^n)$ generated over $\mathbb{C}$ by functions $t \to e^{i\langle\lambda,t\rangle}$ with $\lambda \in \mathbb{Q}$.

3.2. **Definition of holomorphic $\mathfrak{a}$-functions on complex $\mathfrak{a}$-submanifolds in terms of currents.** There is an equivalent definition of holomorphic $\mathfrak{a}$-functions on a complex $\mathfrak{a}$-submanifold Z (cf. Definition 2.8) in terms of currents. Namely, let $\mathfrak{a}$ be self-adjoint, then a function $f \in \mathcal{O}(Z)$ on a complex $\mathfrak{a}$-submanifold $Z \subset X$ is a holomorphic $\mathfrak{a}$-function if and only if it is bounded on subsets $Z \cap p^{-1}(U_0)$, $U_0 \Subset X_0$, and the corresponding current

$$(f,\varphi) := \int_Z f\varphi, \quad \varphi \in \Lambda_c^{k,k}(X), \quad k := \mathrm{codim}_{\mathbb{C}} Z, \tag{3.8}$$

is an $\mathfrak{a}$–current; the latter means that for any φ the function $G \ni g \to \big(f, \varphi_g\big)$ belongs to algebra $\mathfrak{a}$. Here $\varphi_g(x) := \varphi(g \cdot x)$ $(x \in X)$.

In the setting of Example 1.1 (holomorphic almost periodic functions on tube domains) the almost periodic currents were studied, e.g., in [FRR2] (see further references therein).

3.3. **Holomorphic almost periodic functions on coverings of complex manifolds.** The elements of algebra $\mathcal{O}_{AP}(X)$, where $X \to X_0$ is a regular covering as in the Introduction, are called *holomorphic almost periodic functions* (cf. Section 3.1(2) for the definition of algebra $AP(G)$). Equivalently, a function $f \in \mathcal{O}(X)$ is called holomorphic almost periodic if each G-orbit in X has a neighbourhood U that is invariant with respect to the (left) action of G, such that the family of translates $\{z \to f(g \cdot z), z \in U\}_{g \in G}$ is relatively compact in the topology of uniform convergence on U (see [BrK1] for the proof of equivalence).

This is a variant of definition in [We], where G is taken to be the group of all biholomorphic automorphisms of the complex manifold X (an interesting result in [Ves] states that on Siegel domains of the second kind there are no non-constant holomorphic almost periodic functions in the sense of [We], although on Siegel domains of the first kind (i.e., on tube domains in $\mathbb{C}^n$) the holomorphic almost periodic functions even separate points).

For instance, let X_0 be a non-compact Riemann surface, $p : X \to X_0$ be a regular covering with a maximally almost periodic deck transformation group G (for instance, X_0 is hyperbolic, then $X = \mathbb{D}$ is its universal covering, and $G = \pi_1(X_0)$ is a free (not necessarily finitely generated) group); the functions in $\mathcal{O}_{AP}(X)$ arise, e.g., as linear combinations over $\mathbb{C}$ of matrix entries of fundamental solutions of certain linear differential equations on X (see Section 3.7(2) for details).

The algebra $\mathcal{O}_{AP}(X)$ has a number of interesting properties. In particular, we can refine assertion (2) of Theorem 2.14 as follows: if $f \in \mathcal{O}(X)$, $|f| \in C_{AP}(X)$, then E_f is equivalent to a divisor in $\mathrm{Div}_{AP}(X)$.

We also have

Proposition 3.1. *Let $E \in \mathrm{Div}_{AP}(X)$. Then the integration current T_E of divisor E is almost periodic (AP-) (cf. Section 3.2). Conversely, if the integration current T_E of a divisor $E \in$* $\mathrm{Div}(X)$ *is almost periodic, then E is equivalent to an AP-divisor.*

Recall that a complex manifold X_0 is called *ultraliouville* if there are no non-constant bounded continuous plurisubharmonic functions on X_0, e.g. connected compact complex manifolds and their Zariski open subsets. We say that the covering $p : X \to X_0$ has the $\mathcal{O}_{\mathfrak{a}}$-Liouville property if $\mathcal{O}_{\mathfrak{a}}(X)$ does not contain non-constant bounded functions. According to [Lin], if X_0 is ultraliouville and G is virtually nilpotent (i.e., contains a nilpotent subgroup of finite index), then X has $\mathcal{O}_{\ell_\infty}$-Liouville property. For the holomorphic almost periodic functions on covering X, Lin's theorem can be refined as follows:

Theorem 3.2 ([BrK1])**.** *Suppose that X_0 is ultraliouville.*

(1) Then X has $\mathcal{O}_{AP}$-Liouville property.

(2) Let $n \geq 2$, $D_0 \Subset X_0$ be a subdomain with a connected piecewise smooth boundary ∂D_0 contained in a Stein open submanifold of X_0, and $D := p^{-1}(D_0)$. Then $X \setminus D$ has $\mathcal{O}_{AP}$-Liouville property.

For instance, consider the universal covering $p : \mathbb{D} \to \mathbb{C} \setminus \{0, 1\}$ of doubly punctured complex plane (here the deck transformation group is free group with two generators). Although there are plenty of non-constant bounded holomorphic functions on $\mathbb{D}$, all bounded holomorphic almost periodic functions on $\mathbb{D}$ corresponding to this covering are constant because $\mathbb{C} \setminus \{0,1\}$ is ultraliouville.

3.4. **Cylindrical $\mathfrak{a}$-divisors.** The $\mathfrak{a}$-principal divisors are contained in a larger class of cylindrical $\mathfrak{a}$-divisors, i.e., the $\mathfrak{a}$-divisors such that in Definition 2.13 we have $U_\alpha = p^{-1}(U_{0,\alpha})$ for some open $U_{0,\alpha} \subset X_0$, and $f_\alpha \in \mathcal{O}_{\mathfrak{a}}(U_\alpha)$.

For some self-adjoint subalgebras $\mathfrak{a}$ cylindrical $\mathfrak{a}$-divisors exhaust $\mathrm{Div}_{\mathfrak{a}}(X)$ up to the equivalence: if the covering dimension of the maximal ideal space $M_{\mathfrak{a}}$ of $\mathfrak{a}$ is zero, then every $\mathfrak{a}$-divisor is equivalent to a cylindrical $\mathfrak{a}$-divisor (see Section 5.8.1). In particular, all ℓ_∞-, $\hat{\ell}_\infty(G)$- (for a residually finite group G), $AP_{\mathbb{Q}}$-divisors (cf. (1) and (3) in Section 3.1) are equivalent to cylindrical divisors (cf. Examples 4.1(3), (4) below). There are, however, non-cylindrical AP-divisors, cf. Section 3.5(4).

For an arbitrary self-adjoint algebra $\mathfrak{a}$, Theorem 2.15 implies that if E is an $\mathfrak{a}$-divisor, then:

(a) If E is not equivalent to a cylindrical $\mathfrak{a}$-divisor, then the projection of $\mathrm{supp}(E)$ to X_0 is everywhere dense (the converse is not true, see Section 3.5(4)).

(b) If there exists a function $f \in \mathcal{O}(U)$, where $U = p^{-1}(U_0)$, $U_0 \subset X_0$ is open, such that $E|_U$ is determined by f, then E is equivalent to a cylindrical divisor.

3.5. **Examples of complex $\mathfrak{a}$-submanifolds.** We assume that subalgebra $\mathfrak{a}$ is self-adjoint.

(1) If $Z_0 \subset X_0$ is a complex submanifold of codimension k, then $Z := p^{-1}(Z_0) \subset X$ is a complex $\mathfrak{a}$-submanifold of codimension k.

(2) The disjoint union of a finite collection of complex $\mathfrak{a}$-submanifolds Z_i of X separated by the functions in $C_{\mathfrak{a}}(X)$ (i.e., for each i there is $f \in C_{\mathfrak{a}}(X)$ such that $f = 1$ on Z_i, and $f = 0$ on Z_j for $j \neq i$) is a complex $\mathfrak{a}$-submanifold.

(3) Let $Z_0 = \{x \in X_0 : f_1(x) = \cdots = f_k(x) = 0\}$ for some $f_i \in \mathcal{O}(X_0)$ $(1 \le i \le k)$ such that the rank of the Jacobian matrix of the map $x \to \big(f_1(x), \dots, f_k(x)\big)$ with respect to some local coordinates on X_0 is maximal on Z_0. Set $Z := p^{-1}(Z_0)$. Further, for an open subset $X_0' \Subset X_0$ and functions $h_1, \dots, h_k \in \mathcal{O}_{\mathfrak{a}}(X)$ we define $X' := p^{-1}(X_0')$, $\delta := \sup_{x \in X} \max_{1 \le i \le k} |h_i(x)|$, and

$$Z_h := \{x \in X' : p^* f_1(x) + h_1(x) = \cdots = p^* f_k(x) + h_k(x) = 0\}.$$

It is not difficult to see that Z_h is a complex $\mathfrak{a}$-submanifold of X' provided that $\delta > 0$ is sufficiently small.

(4) A complex $\mathfrak{a}$-submanifold of X is called *cylindrical* if each open set V in Definitions 2.6 has form $V = p^{-1}(V_0)$ for some open $V_0 \subset X_0$.

In [BrK1] we have constructed a non-cylindrical $\mathfrak{a}$-hypersurface in X in the case $\mathfrak{a} = AP(\mathbb{Z})$ (cf. Section 3.3) and $p : X \to X_0$ is a regular covering of a Riemann surface X_0 having deck transformation group $\mathbb{Z}$. We assume that X_0 has finite type and is a relatively compact subdomain of a larger (non-compact) Riemann surface $\tilde{X}_0$ whose fundamental group satisfies $\pi_1(\tilde{X}_0) \cong \pi_1(X_0)$ (e.g., the covering of Example 1.1 with $n = 1$, i.e., a complex strip covering an annulus, is a regular covering of this form).

Let us briefly describe this construction.

The covering X of X_0 admits an injective holomorphic map into a holomorphic fibre bundle over X_0 having fibre $(\mathbb{C}^*)^2$, $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, defined as follows. First, note that the regular covering $p : X \to X_0$ admits presentation as a principal fibre bundle with fibre $\mathbb{Z}$, see (2.5). We choose two characters $\chi_1, \chi_2 : \mathbb{Z} \to \mathbb{S}^1 \cong \mathbb{R}/(2\pi\mathbb{Z})$ such that the homomorphism $(\chi_1, \chi_2) : \mathbb{Z} \to \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ is an embedding with dense image. Consider the fibre bundle $b_{\mathbb{T}^2} X$ over X_0 with fibre $\mathbb{T}^2$ associated with the principal fibre bundle $p : X \to X_0$ via the homomorphism

(χ_1,χ_2). The bundle $b_{\mathbb{T}^2}X$ is embedded into a holomorphic fibre bundle $b_{(\mathbb{C}^*)^2}X$ with fibre $(\mathbb{C}^*)^2$ associated with the composite of the embedding homomorphism $\mathbb{T}^2\hookrightarrow(\mathbb{C}^*)^2$ and (χ_1,χ_2). Now, the covering X of X_0 admits an injective C^∞ map into $b_{\mathbb{T}^2}X$ with dense image and the composite of this map with the embedding of $b_{\mathbb{T}^2}X$ into $b_{(\mathbb{C}^*)^2}X$ is an injective holomorphic map $X\to b_{(\mathbb{C}^*)^2}X$. Further, the bundle $b_{(\mathbb{C}^*)^2}X$ admits a holomorphic trivialization $\eta: b_{(\mathbb{C}^*)^2}X\to X_0\times(\mathbb{C}^*)^2$. We choose $\chi_1(1)$ and $\chi_2(1)$ so close to $1\in\mathbb{S}^1$ that the image $\eta(b_{\mathbb{T}^2}X)\subset X_0\times(\mathbb{C}^*)^2$ is sufficiently close to $X_0\times\mathbb{T}^2$. Thus identifying X (by means of holomorphic injection $X\hookrightarrow b_{(\mathbb{C}^*)^2}X\stackrel{\eta}{\to}X_0\times(\mathbb{C}^*)^2$) with a subset of $X_0\times(\mathbb{C}^*)^2$, we obtain that X is sufficiently close to $X_0\times\mathbb{T}^2$. Next, we construct a smooth complex hypersurface in $X_0\times(\mathbb{C}^*)^2$ such that in each cylindrical coordinate chart $U_0\times(\mathbb{C}^*)^2$ on $X_0\times(\mathbb{C}^*)^2$ for $U_0\Subset X_0$ simply connected it cannot be determined as the set of zeros of a holomorphic function on $U_0\times(\mathbb{C}^*)^2$. Intersecting this hypersurface with X we obtain a non-cylindrical almost periodic hypersurface in X. (To construct such a hypersurface in $X_0\times(\mathbb{C}^*)^2$, we determine a smooth divisor in $(\mathbb{C}^*)^2$ that has a non-zero Chern class (i.e., it cannot be given by a holomorphic function on $(\mathbb{C}^*)^2$), and whose support intersects the real torus $\mathbb{T}^2\subset(\mathbb{C}^*)^2$ transversely. Then we take the pullback of this divisor with respect to the projection $X_0\times(\mathbb{C}^*)^2\to(\mathbb{C}^*)^2$ to get the desired hypersurface.)

3.6. **Examples of spaces $\mathfrak{a}_\iota$.** (1) Let $\mathfrak{a}=\ell_\infty(G)$, I be the collection of all subsets of G ordered by inclusion. Given $\iota\in I$, we define $\mathfrak{a}_\iota$ to be the closed linear subspace spanned by translates $\{R_g(\chi_\iota): g\in G\}$ of the characteristic function χ_ι of subset ι.

(2) Let $\mathfrak{a}=AP(\mathbb{Z}^n)$ (cf. Section 3.1(2)). We can take I to be the collection of all finite subsets of $\mathbb{R}^n/2\pi\mathbb{Z}^n$ ordered by inclusion, and $\mathfrak{a}_\iota(\mathbb{Z}^n):=\text{span}_{\mathbb{C}}\{e^{i\langle\lambda,t\rangle},\lambda\in\iota,\iota\in I,t\in\mathbb{Z}^n\}$ (finite-dimensional spaces). (See also Section 3.3 below.)

We can also consider $\mathfrak{a}=AP_{\mathbb{Q}}(\mathbb{Z}^n)$, the algebra of almost periodic functions on $\mathbb{Z}^n$ having rational spectra (cf. Section 3.1(3)). Here we take I to be the collection of all finite subsets of $\mathbb{Q}^n$ ordered by inclusion, and similar spaces $\mathfrak{a}_\iota(\mathbb{Z}^n)$.

(3) Let $\mathfrak{a}=AP(G)$ (cf. Section 3.1(3)), I consists of finite collections of finite-dimensional irreducible unitary representations of group G. We define $\mathfrak{a}_\iota(G)$, where $\iota=\{\sigma_1,\dots,\sigma_m\}\in I$, to be the linear $\mathbb{C}$-hull of matrix elements $\sigma_k^{ij}\in AP(G)$ of representations $\sigma_k=(\sigma_k^{ij})$, $1\le k\le m$ (finite-dimensional spaces).

3.7. **Approximation of holomorphic almost periodic functions.** (1) Let $\mathcal{O}_0(T)$ be determined by the choice of spaces $\mathfrak{a}_\iota=\mathfrak{a}_\iota(\mathbb{Z}^n)$ ($\iota\in I$) as in Section 3.6(2). Let us show that exponential polynomials (1.1) are dense in $\mathcal{O}_0(T)$.

We denote $e_\lambda(t):=e^{i\langle\lambda,t\rangle}$ ($\lambda\in\mathbb{R}^n/2\pi\mathbb{Z}^n$, $t\in\mathbb{Z}^n$). Clearly, $e_\lambda\in\mathcal{O}_{\{\lambda\}}(T)$. Now, let $\iota=\{\lambda_1,\dots,\lambda_m\}$. Since functions e_{λ_k} ($1\le k\le m$) are linearly independent in $\mathfrak{a}_\iota$, there exist linear projections $p_{\iota,\lambda_k}:\mathfrak{a}_\iota\to\mathfrak{a}_{\{\lambda_k\}}$. Since projections p_{ι,λ_k}, $1\le k\le m$, are invariant with respect to the action of G on itself by right translates, they induce projections $P_{\iota,\lambda_k}:\mathcal{O}_\iota(T)\to\mathcal{O}_{\{\lambda_k\}}(T)$. (The latter can be easily seen, e.g., from the presentation of functions in $\mathcal{O}_{AP}(T)$ as sections of holomorphic Banach vector bundle $C_{AP}X_0$, cf. (2.6), where projections P_{ι,λ_k} become bundle homomorphisms $C_{\mathfrak{a}_\iota}X_0\to C_{\mathfrak{a}_{\{\lambda_k\}}}X_0$.) Therefore, there exist functions $f_{\lambda_k}\in\mathcal{O}_{\{\lambda_k\}}(T)$, $f_{\lambda_k}:=P_{\iota,\lambda_k}(f)$, $1\le k\le m$, such that $f(z)=\sum_{k=1}^m f_{\lambda_k}(z)$, $z\in T$. It is now easy to see that for each f_{λ_k} there exists a function $h_{\lambda_k}\in\mathcal{O}(T_0)$ such that $f_{\lambda_k}/e_{\lambda_k}=p^*h_{\lambda_k}$, hence

$$f(z)=\sum_{k=1}^m(p^*h_{\lambda_k})(z)e^{i\langle\lambda_k,z\rangle},\quad z\in T. \tag{3.9}$$

Since the base T_0 of the covering is a relatively complete Reinhardt domain, functions h_{λ_k} admit expansions into Laurent series (see, e.g., [S])

$$h_k(z) = \sum_{|\alpha|=-\infty}^{\infty} b_\alpha z^\alpha, \quad z \in T_0, \quad b_t \in \mathbb{C},$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multiindex, $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Since $p(z) = \big(e^{iz_1}, \dots, e^{iz_n}\big)$, $z = (z_1, \dots, z_n) \in T$ (cf. Example 1.1), each $p^* h_{\lambda_k}$ admits approximations by finite sums

$$\sum_{|\alpha|=-M}^{M} b_\alpha e^{i\langle \alpha, z\rangle}, \quad z \in T, \tag{3.10}$$

converging uniformly on subsets $p^{-1}(W_0) \subset T$, $W_0 \Subset T_0$. Together with (3.9) this implies that exponential polynomials (1.1) are dense in $\mathcal{O}_0(T)$.

A similar argument shows that the algebra of holomorphic almost periodic functions with rational spectra (i.e. that admit approximation by exponential polynomials (1.1) with $\lambda_k \in \mathbb{Q}^n$) coincides with algebra $\mathcal{O}_{AP_{\mathbb{Q}}}(T)$ (cf. Section 3.1(3)).

(2) Let X_0 be a non-compact Riemann surface, $p : X \to X_0$ be a regular covering with a maximally almost periodic deck transformation group G (for instance, X_0 is hyperbolic, then $X = \mathbb{D}$ is its universal covering, and $G = \pi_1(X_0)$ is a free (not necessarily finitely generated) group). The functions in $\mathcal{O}_{AP}(X)$ (cf. Section 3.3) arise, e.g., as linear combinations over $\mathbb{C}$ of matrix entries of fundamental solutions of certain linear differential equations on X.

Indeed, let $\mathcal{U}_G$ be the set of finite dimensional irreducible unitary representations $\sigma : G \to U_m$ ($m \geq 1$), let I be the collection of finite subsets of $\mathcal{U}_G$ directed by inclusion, and for each $\iota \in I$ let $AP_\iota(G)$ be the (finite-dimensional) subspace generated by matrix elements of the unitary representations $\sigma \in \iota$. Then by Theorem 2.18 the $\mathbb{C}$-linear hull $\mathcal{O}_0(X)$ of spaces $\mathcal{O}_\iota(X)$ is dense in $\mathcal{O}_{\mathfrak{a}}(X)$ (note that for each $\sigma \in \mathcal{U}_G$ the space $\mathcal{O}_{\{\sigma\}}(X)$ is the $\mathbb{C}$-linear hull of coordinates of vector-valued functions f in $\mathcal{O}(X, \mathbb{C}^m)$ having the property that $f(g \cdot x) = \sigma(g) f(x)$ for all $g \in G$, $x \in X$). Now, a unitary representation $\sigma : G \to U_m$, $m \geq 1$, can be obtained as the monodromy of the system $dF = \omega F$ on X_0, where ω is a holomorphic 1-form on X_0 with values in the space of $m \times m$ complex matrices $M_m(\mathbb{C})$ (see, e.g., [Fo]). In particular, the system $dF = (p^*\omega)F$ on X admits a global solution $F \in \mathcal{O}\big(X, GL_m(\mathbb{C})\big)$ such that $F \circ g^{-1} = F\sigma(g)$ ($g \in G$). By definition, a linear combination of matrix entries of F is an element of $\mathcal{O}_{AP}(X)$.

3.8. **Approximation property.** Recall that a (complex) Banach space B is said to have the approximation property if for every compact set $K \subset B$ and every $\varepsilon > 0$ there is a bounded operator $T = T_{\varepsilon,K} \in \mathcal{L}(B, B)$ of finite rank so that

$$\|Tx - x\|_B < \varepsilon \quad \text{for every } x \in K.$$

For example, space $AP(G)$ of almost periodic functions on a group G (cf. Section 3.1(2)) has the approximation property with (approximation) operators T in $\mathcal{L}(AP(G), AP_0(G))$ (see, e.g., argument in [Sh]).

Suppose that

(1) spaces $\mathfrak{a}_\iota$, $\iota \in I$, are finite-dimensional, and

(2) space $\mathfrak{a}$ has the approximation property with approximation operators $S \in \mathcal{L}(\mathfrak{a}, \mathfrak{a}_0)$ invariant with respect to the action G on $\mathfrak{a}$ by right translates, i.e., $S(f) = S\big(R_g(f)\big)$ for all $f \in \mathfrak{a}$, $g \in G$,

One can show that if X_0 is a Stein manfiold, $D_0 \subset X_0$ is a strictly pseudoconvex domain, then the Banach space of bounded holomorphic $\mathfrak{a}$-functions $\mathcal{A}_{\mathfrak{a}}(D) := \mathcal{O}_{\mathfrak{a}}(D) \cap C_{\mathfrak{a}}(\bar{D})$ on $D := p^{-1}(D_0)$ (cf. Introduction for notation) has the approximation property with approximation operators in $\mathcal{L}\big(\mathcal{A}_{\mathfrak{a}}(D), \mathcal{A}_0(D)\big)$ (here $\mathcal{A}_0(D)$ is defined similarly to $\mathcal{O}_0(D)$).

4. Fibrewise compactification of a covering

The proofs of results of Section 2 use construction of the fibrewise compactification $c_{\mathfrak{a}}X$ of covering $p : X \to X_0$, introduced in Section 2.1, which we now describe in greater details.

4.1. **Compactification of deck transformation group G.** The (right) action of group G on itself by right multiplication extends to the right action of G on $M_{\mathfrak{a}}$ by the formula

$$\hat{R}_g(\eta)(f) := \eta(R_g(f)), \quad \eta \in M_{\mathfrak{a}}, \quad f \in \mathfrak{a}, \quad g \in G.$$

Then

$$\hat{R}_g(j(t)) = j(tg), \quad t, g \in G. \tag{4.11}$$

We denote by $\mathfrak{Q}$ the basis of topology of $\hat{G}_{\mathfrak{a}}$ consisting of sets of the form

$$\left\{ \eta \in \hat{G}_{\mathfrak{a}} : \max_{1 \le i \le m} |h_i(\eta) - h_i(\eta_0)| < \varepsilon \right\}, \tag{4.12}$$

where $\eta_0 \in \hat{G}_{\mathfrak{a}}$, $h_1, \dots, h_m \in C(\hat{G}_{\mathfrak{a}})$, and $\varepsilon > 0$.

Example 4.1. (1) Let $\mathfrak{a} := c_0(G)$ (cf. Example 1.3(2)). Then $\hat{G}_{c_0}$ is the one-point compactification of G.

(2) Let $\mathfrak{a} = AP(G)$ (cf. Section 3.1(2)). Then $\hat{G}_{AP}$ is homeomorphic to a compact group bG, called Bohr compactification of G, is uniquely determined by the universal property: there exists a (continuous) homomorphism $\mu : G \to bG$ such that any homomorphism $\nu : G \to C$ to a compact group C factors through μ: there is homomorphism $\nu_0 : bG \to C$ with the property $\nu = \nu_0 \mu$.

Applying the universal property to unitary groups $H := U_n$, $n \ge 1$, we obtain that group G is maximally almost periodic (cf. Section 3.1(2)) if and only if μ is an embedding.

The universal property implies that there exists a bijection between sets of finite-dimensional irreducible unitary representations of G and bG. It turn, the Peter-Weyl theorem for $C(bG)$ and von Neumann's approximation theorem for $AP(G)$ (cf. Section 3.1(2)) imply that $AP(G) \cong C(bG)$. Therefore, bG is homeomorphic to the maximal ideal space of algebra $AP(G)$, and $\mu(G)$ is dense in bG. Under this homeomorphism, the set $j(G)$ is identified with the subgroup $\mu(G) \subset bG$ (we also denote $\mu(G)$ by $G \subset bG$), so that the action of G on $\hat{G}_{AP}$ coincides with the action of G on bG by right translations.

By Peter-Weil theorem group bG can be presented as the inverse limit of an inverse system of finite-dimensional compact Lie groups. In particular, the Bohr compactification $b\mathbb{Z}$ of integers $\mathbb{Z}$ is the inverse limit of a family of compact Abelian Lie groups $\mathbb{T}^k \times \oplus_{l=1}^m \mathbb{Z}/(n_l\mathbb{Z})$, $k, m, n_l \in \mathbb{N}$, where $\mathbb{T}^k := (\mathbb{S}^1)^k$ is the real k-torus. It follows that $b\mathbb{Z}$ is disconnected and has infinite covering dimension. The projections (homomorphisms) $b\mathbb{Z} \to \mathbb{T}^k \times \oplus_{l=1}^m \mathbb{Z}/(n_l\mathbb{Z})$ are defined by finite families of characters $\mathbb{Z} \to \mathbb{S}^1$. For instance, let λ_1, $\lambda_2 \in \mathbb{R} \setminus \mathbb{Q}$ be linearly independent over $\mathbb{Q}$ and $\chi_{\lambda_i} : \mathbb{Z} \to \mathbb{S}^1$, $\chi_{\lambda_i}(n) := e^{2\pi i \lambda_i n}$, $i = 1, 2$, be the corresponding characters. Then the map $(\chi_{\lambda_1}, \chi_{\lambda_2}) : \mathbb{Z} \to \mathbb{T}^2$ is extended by continuity to a continuous surjective homomorphism $b\mathbb{Z} \to \mathbb{T}^2$. If λ_1, λ_2 are linearly dependent over $\mathbb{Q}$, then the corresponding extended homomorphism has image in $\mathbb{T}^2$ isomorphic to $\mathbb{S}^1 \times \mathbb{Z}/(m\mathbb{Z})$ for some $m \in \mathbb{N}$.

(3) Let $\mathfrak{a} = AP_{\mathbb{Q}}(\mathbb{Z}^n)$ (cf. Section 3.1(3)). Then $\hat{G}_{AP_{\mathbb{Q}}}$ is homeomorphic to the profinite completion of group $\mathbb{Z}^n$, it admits presentation as the inverse limit of groups $\oplus_{l=1}^m \mathbb{Z}/(n_l\mathbb{Z})$, $m, n_l \in \mathbb{N}$. It follows that the covering dimension of $\hat{G}_{AP_{\mathbb{Q}}}$ is zero.

(4) Let $\mathfrak{a} = \ell_\infty(G)$ (cf. Example 1.3(1)). Then $\hat{G}_{\ell_\infty} \cong \beta G$, the Stone–Čech compactification of group G. The covering dimension of $\hat{G}_{\ell_\infty}$ is zero (see, e.g., [Hz]).

4.2. **Structure of fibrewise compactification.** Recall that $c_{\mathfrak{a}}X$ was defined as the fibre bundle associated to the regular covering $p : X \to X_0$ (viewed as a principal bundle) having fibre $\hat{G}_{\mathfrak{a}}$ (cf. Definition 2.1). Using the construction of $p : X \to X_0$ as a principal fibre bundle, cf. (2.5), we obtain that $\bar{p} : c_{\mathfrak{a}}X \to X_0$ is obtained from the disjoint union $\sqcup_\gamma U_\gamma \times \hat{G}_{\mathfrak{a}}$ by identification

$$U_\gamma \times \hat{G}_{\mathfrak{a}} \ni (x, \omega) \sim (x, \hat{R}_{c_{\delta\gamma}(x)}(\omega)) \in U_\delta \times \hat{G}_{\mathfrak{a}}, \quad \text{for all } x \in U_\gamma \cap U_\delta,$$

where $\bar{p}$ is induced by projections $U_\gamma \times \hat{G}_{\mathfrak{a}} \to U_\gamma$.

The fibrewise compactification $c_{\mathfrak{a}}X$ (cf. Definition 2.1) is a paracompact Hausdorff space, as a fibre bundle with paracompact base and compact fibre. It follows that $c_{\mathfrak{a}}X$ is normal.

Further, as a set, $c_{\mathfrak{a}}X$ admits presentation as the disjoint union of connected complex manifolds, each being a covering of X_0. Indeed, let $\Upsilon := \hat{G}_{\mathfrak{a}}/G$ (the set of orbits elements of $\hat{G}_{\mathfrak{a}}$ by the action of G); since any orbit $H \in \Upsilon$ is closed with respect to the action of G, we may consider the associated fibre bundle $p_H : X_H \to X_0$. We assume that H is endowed with discrete topology. Then $p_H : X_H \to X_0$ is a covering of X_0 (in general non-regular). Since X is connected, X is a covering of X_H, the complex manifold X_H is connected as well. It follows that as a set $c_{\mathfrak{a}}X = \bigsqcup_{H\in\Upsilon} X_H$. For each $H \in \Upsilon$ we have an injective map $\iota_H : X_H \hookrightarrow c_{\mathfrak{a}}X$ determined by inclusion $H \hookrightarrow \hat{G}_{\mathfrak{a}}$. We denote $\hat{X}_H := \iota_H(X_H)$. In view of (4.11), we have $j(G) \in \Upsilon$. Hence, if j is injective, then $X = X_{j(G)}$ and $\iota = \iota_{j(G)}$ (cf. (2.3)).

Example 4.2. $\hat{G}_{c_0}$ is the one-point compactification of G, and the action of G on $\hat{G}_{c_0}$ fixes the 'point at infinity', so here $\Upsilon = \{G, `\infty'\}$. It follows that as a set the fibrewise compactification $c_{c_0}X$ is the disjoint union of X and X_0.

Example 4.3. Let $\mathfrak{a} = AP(G)$. In what follows, we assume that $\hat{G}_{AP}$ is endowed with the group structure of Bohr compactification bG, cf. Example 4.1(2).

Recall that G is a subgroup of bG and acts on bG by right translations. Therefore, every orbit $H \in \Upsilon$ (cf. Section 4.2) is a right coset of G in bG, $X_H = X$ for all $H \in \Upsilon$, and each set $\hat{X}_H$ is dense in $c_{AP}X$.

The fibre bundle $c_{AP}X$ can be presented as the inverse limit of smooth fibre bundles. Indeed, by Peter-Weil theorem bG can be presented as an inverse limit of an inverse system of finite-dimensional compact Lie groups $\{C_s : s \in S\}$ (cf. Example 4.1(2)), where $\pi_s : bG \to C_s$ denote the corresponding projection homomorphisms. These Lie groups can be taken as the fibres of the required smooth fibre bundles:

$$U_\delta \times C_s \ni (x, h) \sim (x, h \cdot \pi_s(c_{\delta\gamma}(x))) \in U_\gamma \times C_s \quad \text{for all } x \in U_\gamma \cap U_\delta$$

(cf. (2.5)). (Note that bundle $b_{\mathbb{T}^2}X$ constructed in Section 3.5(4) (there $G = \mathbb{Z}$) is of this form.)

Example 4.4. Let $\mathfrak{a} = AP_{\mathbb{Q}}(\mathbb{Z}^n)$. Since the covering dimension of $\hat{G}_{AP_{\mathbb{Q}}}$ is zero (cf. Example 4.1(3)), the covering dimension of $c_{AP_{\mathbb{Q}}}X$ is equal to $\dim_{\mathbb{R}} X_0$.

Example 4.5. Let $\mathfrak{a} = \ell_\infty(G)$. Then $\hat{G}_{\ell_\infty} \cong \beta G$, the Stone–Čech compactification of group G. Since the covering dimension of $\hat{G}_{\ell_\infty}$ is zero, the covering dimension of $c_{\ell_\infty}X$ coincides with the real dimension of X_0.

It is easy to see that $c_{\ell_\infty}X$ is the maximal fibrewise compactification of covering X, in the sense that if $\mathfrak{a}$ is some other algebra, then there is a surjective bundle morphisms $c_{\ell_\infty}X \to c_{\mathfrak{a}}X$. Indeed, there is a surjective map $\kappa : \hat{G}_{\ell_\infty} \to \hat{G}_{\mathfrak{a}}$ adjoint to inclusion $\mathfrak{a} \hookrightarrow \ell_\infty(G)$; since this map is equivariant with respect to the action of G, the existence of surjective bundle morphism follows. In the proofs we will need the fact that using the axiom of choice we can find a right inverse $\lambda : \hat{G}_{\mathfrak{a}} \to \hat{G}_{\ell_\infty}$ to κ, i.e., $\kappa \circ \lambda = \mathrm{Id}$.

4.3. **Holomorphic functions on $c_{\mathfrak{a}}X$.** A function $f \in C(U)$ on an open set $U \subset c_{\mathfrak{a}}X$ is called holomorphic if $\iota^* f$ is holomorphic on $\iota^{-1}(U) \subset X$ in the usual sense.

Let $\mathcal{O}(U)$ denote the algebra of holomorphic functions on U, endowed with the topology of uniform convergence on compact subsets of U. It is immediate that a function $f \in C(c_{\mathfrak{a}}X)$ is in $\mathcal{O}(c_{\mathfrak{a}}X)$ if and only if each point in $c_{\mathfrak{a}}X$ has a neighbourhood U such that $f|_U \in \mathcal{O}(U)$.

The next proposition gives another characterization of functions in $\mathcal{O}_{\mathfrak{a}}(X)$.

Proposition 4.6. *The following is true:*

(1) *A function f in $C_{\mathfrak{a}}(X)$ determines a unique function $\hat{f}$ in $C(c_{\mathfrak{a}}X)$ such that $\iota^*\hat{f} = f$; we have $f \in \mathcal{O}_{\mathfrak{a}}(X)$ if and only if $\hat{f} \in \mathcal{O}(c_{\mathfrak{a}}X)$. Thus, there are continuous embeddings $C_{\mathfrak{a}}(X) \hookrightarrow C(c_{\mathfrak{a}}X)$, $\mathcal{O}_{\mathfrak{a}}(X) \hookrightarrow \mathcal{O}(c_{\mathfrak{a}}X)$.*
(2) *If $\mathfrak{a}$ is self-adjoint, then $C_{\mathfrak{a}}(X) \cong C(c_{\mathfrak{a}}X)$ and $\mathcal{O}_{\mathfrak{a}}(X) \cong \mathcal{O}(c_{\mathfrak{a}}X)$.*

4.4. **Holomorphic maps.** We now introduce the notion of a holomorphic map between complex manifolds and/or open subsets of $c_{\mathfrak{a}}X$.

Let $U_0 \subset X_0$, $K \subset \hat{G}_{\mathfrak{a}}$ be open. We say that a function $f \in C(U_0 \times K)$ is holomorphic on $U_0 \times K$ if $f(\cdot, j(g)) \in \mathcal{O}(U_0)$ for all $g \in j^{-1}(K) \subset G$.

We denote the algebra of holomorphic functions on $U_0 \times K$ by $\mathcal{O}(U_0 \times K)$.

Let M_i $(i = 1, 2)$ be either a complex manifold, or open subset of $c_{\mathfrak{a}}X$, or a set $U_0 \times K$ as above. Let $\mathcal{O}_{M_i}$ be the sheaf of germs of holomorphic functions on M_i.

Definition 4.7. A map $F \in C(M_1, M_2)$ is called holomorphic if $F^*\mathcal{O}_{M_2} \subset \mathcal{O}_{M_1}$.

We denote the collection of holomorphic maps $M_1 \to M_2$ by $\mathcal{O}(M_1, M_2)$. If a holomorphic map in $\mathcal{O}(M_1, M_2)$ has inverse in $\mathcal{O}(M_2, M_1)$, we call it a biholomorphism.

Remark 4.8. Let $\mathfrak{a}$ be self-adjoint. One can show that the holomorphic structure of $c_{\mathfrak{a}}X$ is concentrated only in 'horizontal layers' $\hat{X}_H \subset c_{\mathfrak{a}}X$ $(H \in \Upsilon)$, i.e. for any connected complex manifold M and any map $G \in \mathcal{O}(M, c_{\mathfrak{a}}X)$ there is $H \in \Upsilon$ such that $G(M) \subset \hat{X}_H$.

4.5. **Coordinate charts.** Over each simply connected open subset $U_0 \subset X_0$ there exists a biholomorphic trivialization $\psi = \psi_{U_0} : p^{-1}(U_0) \to U_0 \times G$ of covering $p : X \to X_0$, which is a morphism of fibre bundles with fibres G. In what follows, we fix some system of biholomorphic trivializations of $p : X \to X_0$.

There exists a biholomorphic trivialization $\bar{\psi} = \bar{\psi}_{U_0} : \bar{p}^{-1}(U_0) \to U_0 \times \hat{G}_{\mathfrak{a}}$ of bundle $c_{\mathfrak{a}}X$, which is a morphism of fibre bundles with fibre $\hat{G}_{\mathfrak{a}}$ such that $\bar{\psi} : \bar{p}^{-1}(U_0) \to U_0 \times j(G)$. Thus, we have a commutative diagram

$$\begin{array}{ccc} p^{-1}(U_0) & \xrightarrow{\ \iota\ } & \bar{p}^{-1}(U_0) \\ \big\downarrow{\scriptstyle \psi} & & \big\downarrow{\scriptstyle \tilde{\psi}} \\ U_0 \times G & \xrightarrow{\mathrm{Id}\times j} & U_0 \times \hat{G}_{\mathfrak{a}} \end{array}$$

Notation. For a given subset $S \subset G$ we denote

$$\Pi(U_0, S) := \psi^{-1}(U_0 \times S) \tag{4.13}$$

and identify $\Pi(U_0, S)$ with $U_0 \times S$ where appropriate. Note that $\Pi(U_0, G) = p^{-1}(U_0)$.

For a subset $K \subset \hat{G}_{\mathfrak{a}}$ we denote

$$\hat{\Pi}(U_0, K) \ \big(= \hat{\Pi}_{\mathfrak{a}}(U_0, K)\big) := \bar{\psi}^{-1}(U_0 \times K). \tag{4.14}$$

A subset of the form $\hat{\Pi}(U_0, K)$ will be called a *coordinate chart* for $c_{\mathfrak{a}}X$. Similarly, we idenitfy $\hat{\Pi}(U_0, K)$ with $U_0 \times K$. Clearly, if $K \subset \hat{G}_{\mathfrak{a}}$ is open, then $\mathcal{O}(\hat{\Pi}(U_0, K)) \cong \mathcal{O}(U_0 \times K)$.

4.6. **Basis of topology.** It is easy to see that

$$\mathfrak{B} := \{\hat{\Pi}(V_0, L) \subset c_{\mathfrak{a}}X : V_0 \text{ is open simply connected in } X_0 \text{ and } L \in \mathfrak{Q}\}. \tag{4.15}$$

is a basis of topology of $c_{\mathfrak{a}}X$ (cf. (4.12)).

4.7. **More on coherent sheaves.** In Section 2.1 we introduced the notions of analytic and coherent sheaves on $c_{\mathfrak{a}}X$.

An analytic sheaf $\mathcal{A}$ on $c_{\mathcal{A}}X$ is called a *Fréchet sheaf* if for each open set $U \in \mathfrak{B}$ (cf. (4.15)) the module of sections $\Gamma(U, \mathcal{A})$ of $\mathcal{A}$ over U is endowed with a topology of Fréchet space.

Proposition 4.9. *Every coherent sheaf can be turned in a unique way into a Fréchet sheaf so that the following conditions are satisfied:*

(1) If $\mathcal{A}$ is a coherent subsheaf of $\mathcal{O}$ then for any open subset $U \in \mathfrak{B}$ the module of sections $\Gamma(U, \mathcal{A})$ has the topology of uniform convergence on compact subsets of U, for all k.

(2) If $\mathcal{A}, \mathcal{B}$ are coherent sheaves on $c_{\mathfrak{a}}X$, then for any $U \in \mathfrak{B}$ the spaces $\Gamma(U, \mathcal{A})$, $\Gamma(U, \mathcal{B})$ are Fréchet spaces spaces, and any analytic homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ in continuous in the sense that the homomorphisms of sections of $\mathcal{A}$ and $\mathcal{B}$ over sets $U \in \mathfrak{B}$ induced by φ are continuous..

The topology on $\Gamma(U, \mathcal{A})$ can be defined by a family of semi-norms

$$\|f\|_{V_k} := \inf\left\{\sup_{x \in V_k} |h(x)| : h \in \Gamma(V_k, \mathcal{O}^{m_1}),\ f = (\varphi_0)_*(h)\right\},$$

where $(\varphi_0)_*$ is the homomorphism of sections induced by φ_0 in (2.4), and open sets $V_k \in \mathfrak{B}$ are such that $V_k \Subset V_{k+1} \Subset U$ for all k, and $U = \cup_k V_k$ (see Lemma 5.10.4(2) below).

We denote by ${}_x\mathcal{B}$ the stalk of an analytic sheaf $\mathcal{B}$ at $x \in c_{\mathfrak{a}}X$.

Our main result concerning coherent sheaves on $c_{\mathfrak{a}}X$ is the following

Theorem 4.10. *Let X_0 be a Stein manifold, $\mathcal{A}$ a coherent analytic sheaf on $c_{\mathfrak{a}}X$. Then the following is true:*

(A) For every $x \in c_{\mathfrak{a}}X$ the stalk ${}_x\mathcal{A}$ is generated by the global sections in $\Gamma(c_{\mathfrak{a}}X, \mathcal{A})$ as an ${}_x\mathcal{O}$-module.

(B) For all $i \geq 1$ the Čech cohomology groups $H^i(c_{\mathfrak{a}}X, \mathcal{A}) = 0$.

(C) (Runge-type) Suppose that $Y_0 \Subset X_0$, $\hat{Y} \subset c_{\mathfrak{a}}X$ are open and such that either (1) Y_0 is holomorphically convex in X_0 and $\hat{Y} = \bar{p}^{-1}(Y_0)$, or (2) Y_0 is holomorphically convex in X_0 and is contained in a simply connected open subset of X_0, and $\hat{Y} = \hat{\Pi}(Y_0, K)$ for some $K \in \mathfrak{Q}$ (cf. 4.12).

Then the restriction map $\Gamma(c_{\mathfrak{a}}X, \mathcal{A}) \to \Gamma(\hat{Y}, \mathcal{A})$ has dense image.

Statements (A) and (B) of Theorem 4.10 coincide with Theorems 2.3 and 2.4, respectively.

4.8. **Maximal ideal space.** We now relate the fibrewise compactification $c_{\mathfrak{a}}X$ of covering X with the maximal ideal space M_X of algebra $\mathcal{O}_{\mathfrak{a}}(X)$, i.e., the space of non-zero continuous homomorphisms $\mathcal{O}_{\mathfrak{a}}(X) \to \mathbb{C}$ endowed with weak* topology (of $\mathcal{O}_{\mathfrak{a}}(X)^*$).

Theorem 4.11. *Suppose that algebra $\mathfrak{a}$ is self-adjoint, and X_0 is a Stein manifold. Then M_X is homeomorphic to $c_{\mathfrak{a}}X$.*

Since $\iota(X)$ is dense in $c_{\mathfrak{a}}X$, and the natural mapping of X into M_X, sending each point of X to its point evaluation homomorphism, coincides with ι under the homeomorphism of Theorem 4.11, we obtain the following corona-type theorem.

Corollary 4.12. *If $\mathfrak{a}$ is self-adjoint, and X_0 is Stein, then X is dense in M_X.*

5. Proofs

5.1. Preliminaries

5.1.1. **Čech cohomology.** For a topological space A and a sheaf of Abelian groups $\mathcal{S}$ on A, let us denote by $\Gamma(A, \mathcal{S})$ the Abelian group of continuous sections of $\mathcal{S}$ over A.

Let $\mathcal{U}$ be an open cover of A. We denote by $\mathcal{C}^i(\mathcal{U}, \mathcal{S})$ the space of Čech i-cochains with values in $\mathcal{S}$, by $\delta : \mathcal{C}^i(\mathcal{U}, \mathcal{S}) \to \mathcal{C}^{i+1}(\mathcal{U}, \mathcal{R})$ the Čech coboundary operator (for detailed definition see, e.g., [GR]), by $\mathcal{Z}^i(\mathcal{U}, \mathcal{S}) := \{\sigma \in \mathcal{C}^i(\mathcal{U}, \mathcal{S}) : \delta\sigma = 0\}$ the space of i-cocycles, and by $\mathcal{B}^i(\mathcal{U}, \mathcal{S}) := \{\sigma \in \mathcal{Z}^i(\mathcal{U}, \mathcal{S}) : \sigma = \delta(\eta), \eta \in \mathcal{C}^{i-1}(\mathcal{U}, \mathcal{S})\}$ the space of i-coboundaries. The Čech cohomology groups $H^i(\mathcal{U}, S)$, $i \geq 0$, are defined by

$$H^i(\mathcal{U}, S) := \mathcal{Z}^i(\mathcal{U}, \mathcal{S})/\mathcal{B}^i(\mathcal{U}, \mathcal{S}), \quad i \geq 1,$$

and $H^0(\mathcal{U}, \mathcal{S}) := \Gamma(\mathcal{U}, \mathcal{S})$.

5.1.2. **$\bar{\partial}$-equation.** Let B be a complex Banach space, $D_0 \subset X_0$ be a strictly pseudoconvex domain. In what follows, we assume that we have fixed a system of local coordinates on D_0. Let $\{W_{0,i}\}_{i \geq 1}$ be the cover of D_0 by coordinate patches. By $\Lambda_b^{(0,q)}(D_0, B)$, $q \geq 0$, we denote the space of bounded continuous B-valued $(0, q)$-forms on D_0 endowed with norm

$$\|\omega\|_{D_0} = \|\omega\|_{D_0}^{(0,q)} := \sup_{x \in U_{i,0}, i \geq 1, \alpha} \|\omega_{\alpha,i}(x)\|_B, \tag{1.16}$$

where $\omega_{\alpha,i}$ (α is a multiindex) are the coefficients of form $\omega|_{W_{0,i}} \in \Lambda_b^{(0,q)}(W_{0,i}, B)$ in local coordinates on $W_{0,i}$.

The next lemma follows easily from the results in [HL] (proved for $B = \mathbb{C}$), as all integral presentations and estimates are preserved when passing to the case of Banach-valued forms.

Lemma 5.1.1. *There exists a bounded linear operator*

$$R_{D_0,B} \in \mathcal{L}\left(\Lambda_b^{(0,q)}(D_0, B), \Lambda_b^{(0,q-1)}(D_0, B)\right), \quad q \geq 1,$$

such that if $\omega \in \Lambda_b^{(0,q)}(D_0, B)$ is C^∞ and satisfies $\bar{\partial}\omega = 0$ on D_0, then $\bar{\partial} R_{D_0,B}\omega = \omega$ on D_0.

5.2. Proof of Theorem 1.2

Let $T^s = \mathbb{R}^n + i\bar{\Omega}^s \subset \mathbb{C}^n$, where $\bar{\Omega}^s \subset \mathbb{R}^n$ is compact. We will need the following definition.

Definition 5.2.1. A function $f \in C(T^s)$ is called *continuous almost periodic* if the family of its translates $\{T^s \ni z \mapsto f(z+t)\}$, $t \in \mathbb{R}^n$, is relatively compact in $C_b(T^s)$ (space of bounded continuous functions on T^s endowed with sup-norm).

We set

$$p(z) := \left(e^{iz_1}, \dots, e^{iz_n}\right), \quad z = (z_1, \dots, z_n) \in T^s$$

and $T_0^s := p(T^s)$. Thus, we have subalgebra $C_{AP}(T^s)$, where $AP = AP(\mathbb{Z}^n)$, $\mathbb{Z}^n \cong p^{-1}(x_0)$ ($x_0 \in X_0$) (cf. Introduction for corresponding notation and definitions).

For the proof of Theorem 1.2 we only have to show that $APC(T^s) = C_{AP}(T^s)$.

First, let $f \in APC(T^s)$, i.e., for any sequence $\{t_k\} \subset \mathbb{R}^n$ there exists a subsequence of $\{T^s \ni z \mapsto f(z+t_k)\}$ that converges uniformly on T^s. Then f is uniformly continuous on T^s and for every $z_0 \in T^s$ and $\{d_k\} \subset \mathbb{Z}^n$ the family of translates $\{\mathbb{Z}^n \ni g \mapsto f(z_0 + g + d_k)\}$ is relatively compact, hence $f \in C_{AP}(T^s)$.

Now, let $f \in C_{AP}(T^s)$. Let us show that $f \in APC(T^s)$. We fix some sequence $\{t_k\} \subset \mathbb{R}^n$. We have a continuous group homomorphism $\mu : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$. Since $\mathbb{R}^n/\mathbb{Z}^n$ is compact, $\{\mu(s_k)\}$ has a convergent subsequence; without loss of generality we may assume that $\{\mu(s_k)\}$ converges to 0. Hence, there exists a sequence $\{d_k\} \subset \mathbb{Z}^n$ such that $|t_k - d_k| \to 0$ as $k \to \infty$. Since f is uniformly continuous on T^s,

$$|f(z+t_k) - f(z+d_k)| \to 0 \quad \text{uniformly on } T^s \text{ as } k \to \infty.$$

Thus, it suffices to show that $\{T^s \ni z \to f(z+d_k)\}$ has a convergent subsequence. Let $C := \{z = (z_1, \dots, z_n) \in T^s : 0 \le \text{Re}\,(z_i) \le 1, 1 \le i \le n\}$. Since $f \in C_{AP}(T^s)$, for every $z_0 \in C$ the family of translates $\{\mathbb{Z}^n \ni g \mapsto f(z_0 + g + d_k)\}$ is relatively compact in the topology of uniform convergence on $\mathbb{Z}^n$, i.e., there exists a subsequence $\{d_{k_l}\}$ (depending on z_0) such that for every $\varepsilon > 0$ there exists N such that for all $l, m > N$

$$|f(z_0 + g + d_{k_l}) - f(z_0 + g + d_{k_m})| < \frac{\varepsilon}{3} \quad \text{for all } g \in \mathbb{Z}^n.$$

Now, since f is uniformly continuous on T^s, there exists $\delta > 0$ such that

$$|f(z+h) - f(z_0+h)| < \frac{\varepsilon}{3} \quad \text{for all } z \in C, |z - z_0| < \delta \text{ and all } h \in \mathbb{Z}^n.$$

It follows that for all $l, m > N$

$$|f(z + g + d_{k_l}) - f(z + g + d_{k_m})| < \varepsilon \quad \text{for all } z \in C, |z - z_0| < \delta, \quad g \in \mathbb{Z}^n.$$

Since C is compact, we need to consider only finitely many δ-neighbourhoods of points in C that cover C, i.e., passing to a subsequence of $\{d_{k_l}\}$ finitely many times (without loss of generality this is $\{d_{k_l}\}$ itself) we obtain that for all $l, m > N$

$$|f(z + g + d_{k_l}) - f(z + g + d_{k_m})| < \varepsilon \quad \text{for all } z \in C, \quad g \in \mathbb{Z}^n,$$

i.e., since $\{z + g : z \in C, g \in \mathbb{Z}^n\} = T^s$, for all $l, m > N$

$$|f(z + d_{k_l}) - f(z + d_{k_m})| < \varepsilon \quad \text{for all } z \in T^s.$$

The constructed subsequence $\{d_{k_l}\}$ depends on $\varepsilon > 0$. Now, let $\varepsilon_r \to 0+$ as $r \to \infty$, $\varepsilon := \varepsilon_1$. We leave $2N > 1$ first terms of $\{d_{k_l}\}$ unchanged, and for $\{d_{k_l}\}_{l \ge 2N+1}$ set $\varepsilon := \varepsilon_2$, and apply the previous procedure, passing to a subsequence of $\{d_{k_l}\}_{l \ge 2N+1}$, etc. We obtain a subsequence of $\{d_{k_l}\}$ (without loss of generality, this is $\{d_{k_l}\}$ itself) such that for every $\varepsilon_r > 0$ there exists

N_r such that for all $l, m > N_r$ $|f(z + d_{k_l}) - f(z + d_{k_m})| < \varepsilon_r$ for all $z \in T^s$, i.e., the sequence $\{T^s \ni z \mapsto f(z + d_{k_l})\}$ converges uniformly on T^s, as needed.

5.3. Proofs of Propositions 2.11 and 4.6

The proofs of our results are based on the equivalences established in Propositions 2.11 and 4.6, so we prove these propositions first.

Proof of Proposition 2.11. We prove the assertion for holomorphic functions, the proof for continuous functions is analogous.

Let us establish the first isomorphism. It is easy to see that any function $f \in \mathcal{O}_{\mathfrak{a}}(X)$ is locally Lipschitz with respect to the semi-metric d (see Introduction), i.e.,

$$|f(x_1, g) - f(x_2, g)| \le Cd\big((x_1, g), (x_2, g)\big) := Cd_0(x_1, x_2) \tag{3.17}$$

for all (x_1, g), $(x_2, g) \in W_0 \times G \cong p^{-1}(W_0)$, where $W_0 \Subset X_0$ is a simply connected coordinate chart. (Here C depends on d_0 and W_0 only.) We denote $f_{x_0} := f|_{p^{-1}(x_0)} \in \mathfrak{a}$, $x_0 \in X_0$, and define

$$\tilde{f}(x_0) := f_{x_0}, \quad x_0 \in X_0.$$

Then $\tilde{f}$ is a section of bundle $C_{\mathfrak{a}} X_0$. From (3.17) for any linear functional $\varphi \in \mathfrak{a}^*$ we have $\varphi(\tilde{f}(x)(g)) := \varphi(f(x, g)) \in \mathcal{O}(W_0)$, $g \in G$, $x \in W_0 \Subset X_0$, a simply connected coordinate chart, cf. [Lin] for similar arguments. Thus $\tilde{f}$ is a holomorphic section of $C_{\mathfrak{a}} X_0$. Reversing these arguments we obtain that any holomorphic section of $C_{\mathfrak{a}} X_0$ determines an almost periodic holomorphic function on X. The proof of the second isomorphism is similar. □

Proof of Proposition 4.6. Given $f \in \mathcal{O}_{\mathfrak{a}}(X)$, denote $f_{x_0} := f|_{p^{-1}(x_0)}$ and then define $\hat{f}_{x_0} \in C(\hat{G}_{\mathfrak{a}})$ to be the extension of f_{x_0} from $p^{-1}(x_0) \cong G$ to $\bar{p}^{-1}(x_0) \cong \hat{G}_{\mathfrak{a}}$ so that $j^* \hat{f}_{x_0} = f_{x_0}$. The family of the extended functions over points of X_0 determines a function $\hat{f}$ on $c_{\mathfrak{a}} X$ such that $\hat{f}(x) = \hat{f}_{x_0}(x)$ for $x_0 := \bar{p}(x)$. Using a normal family argument one shows that $\hat{f} \in \mathcal{O}(c_{\mathfrak{a}} X)$, see, e.g., [Lin] or [BrK3, Lemma 2.3] for similar results. Clearly, f is such that $\iota^* \hat{f} = f$. Since the algebra homomorphism $\mathfrak{a} \to C(\hat{G}_{\mathfrak{a}})$ is an injection, the constructed homomorphism $i : \mathcal{O}_{\mathfrak{a}}(X) \to \mathcal{O}(c_{\mathfrak{a}} X)$ is an injection too. This completes the proof of the first assertion.

For the proof of the second assertion, suppose that $\mathfrak{a}$ be self-adjoint. Then $\mathfrak{a} \cong C(\hat{G}_{\mathfrak{a}})$, and we can define the inverse homomorphism $i^{-1} : \mathcal{O}(c_{\mathfrak{a}} X) \to \mathcal{O}(X)$ by the formula

$$i(\hat{f}) := \iota^* \hat{f}, \quad \hat{f} \in \mathcal{O}(c_{\mathfrak{a}} X).$$

Since $i^{-1}(\hat{f})|_{p^{-1}(x_0)} = j^*\big(\hat{f}|_{\bar{p}^{-1}(x_0)}\big) \in \mathfrak{a}$, $x_0 \in X_0$, we have $i^{-1}(\hat{f}) \in \mathcal{O}_{\mathfrak{a}}(X)$, i.e., i^{-1} maps $\mathcal{O}(c_{\mathfrak{a}} X)$ into $\mathcal{O}_{\mathfrak{a}}(X)$. □

5.4. Proofs of Theorems 2.16 and 2.17

We will use notation and results of Section 4.5 and Example 4.5. We also introduce the following notation. For a given subset $K \subset G$, we denote by $\hat{K}_{\mathfrak{a}} \subset \hat{G}_{\mathfrak{a}}$ and $\hat{K}_{\ell^\infty} \subset \hat{G}_{\ell_\infty}$ the closures of sets $j_{\mathfrak{a}}(K)$ and $j_{\ell_\infty}(K)$ in $\hat{G}_{\mathfrak{a}}$ and $\hat{G}_{\ell_\infty}$, respectively. We have a commutative diagram

$$
\begin{array}{ccccc}
\Pi(U_0,K) & \xrightarrow{\mathrm{Id}\times j_{\mathfrak{a}}} & \hat{\Pi}_{\mathfrak{a}}(U_0,\hat{K}_{\mathfrak{a}}) & \xrightarrow{\lambda} & \hat{\Pi}_{\ell_\infty}(U_0,\hat{K}_{\ell_\infty}) \\
\uparrow{=} & & \uparrow{\kappa} & \nearrow{=} & \\
\Pi(U_0,K) & \xrightarrow{\mathrm{Id}\times j_{\ell_\infty}} & \hat{\Pi}_{\ell_\infty}(U_0,\hat{K}_{\ell_\infty}) & &
\end{array}
\tag{4.18}
$$

All maps, except possibly for λ, are continuous.

5.4.1. **Proof of Theorem 2.16.** We will need the following results.

Lemma 5.4.1. *There exists a unique function $\hat{f}\in\mathcal{O}\big(\hat{\Pi}_{\mathfrak{a}}(U_0,\hat{K}_{\mathfrak{a}})\big)$ such that*

$$f|_{\Pi(U_0,K)}=(\mathrm{Id}\times j_{\mathfrak{a}})^*\hat{f}. \tag{4.19}$$

Proof of Lemma. Since $f\in\mathcal{O}_{\ell_\infty}(X)$, there exists a function $\tilde{f}\in\mathcal{O}\big(\hat{\Pi}_{\ell_\infty}(U_0,\hat{K}_{\mathfrak{a}})\big)$ such that $f|_{\Pi(U_0,K)}=(\mathrm{Id}\times j_{\ell_\infty})^*\tilde{f}$. We set $\hat{f}:=(\mathrm{Id}\times\lambda)^*\tilde{f}:\hat{\Pi}_{\mathfrak{a}}(U_0,\hat{K}_{\mathfrak{a}})\to\mathbb{C}$. Clearly, (4.19) is satisfied. Identifying $\hat{\Pi}_{\mathfrak{a}}(U_0,\hat{K}_{\mathfrak{a}})$ with $U_0\times\hat{K}_{\mathfrak{a}}$ (cf. (4.14)), we obtain that $\hat{f}(\cdot,\omega)\in\mathcal{O}(U_0)$ for all $\omega\in\hat{K}_{\mathfrak{a}}$. It remains to show that $\hat{f}$ is continuous. By our assumption, there exists a function $F\in C(\hat{\Pi}_{\mathfrak{a}}(Z_0,\hat{K}_{\mathfrak{a}}))$ such that $f|_{\Pi(Z_0,K)}=(\mathrm{Id}\times j_{\mathfrak{a}})^*F$. Since $(\mathrm{Id}\times j_{\mathfrak{a}})\big(\Pi(Z_0,K)\big)$ is dense in $\hat{\Pi}_{\mathfrak{a}}(Z_0,\hat{K}_{\mathfrak{a}})$, and diagram (4.18) is commutative, we have

$$\hat{f}|_{\hat{\Pi}_{\mathfrak{a}}(Z_0,\hat{K}_{\mathfrak{a}})}=F. \tag{4.20}$$

We identify $\hat{\Pi}_{\mathfrak{a}}(U_0,\hat{K}_{\mathfrak{a}})$ with $U_0\times\hat{K}_{\mathfrak{a}}$, and $\hat{\Pi}_{\mathfrak{a}}(Z_0,\hat{K}_{\mathfrak{a}})$ with $Z_0\times\hat{K}_{\mathfrak{a}}$. Suppose that $\hat{f}$ is discontinuous, i.e., there exists a net $\{\omega_\alpha\}\subset\hat{K}_{\mathfrak{a}}$, $\omega_\alpha\to\omega\in\hat{K}_{\mathfrak{a}}$, such that $\hat{f}(\cdot,\omega_\alpha)\not\to\hat{f}(\cdot,\omega)$ in $\mathcal{O}(U_0)$. Using Montel theorem, we obtain that there exists a partial limit of $\{\hat{f}(\cdot,\omega_\alpha)\}$ in $\mathcal{O}(U_0)$ that does not coincide with $\hat{f}(\cdot,\omega)$. However, since $\hat{f}|_{Z_0\times\hat{K}_{\mathfrak{a}}}$ is continuous, cf. (4.20), and Z_0 is a uniqueness set for holomorphic functions in $\mathcal{O}(U_0)$, this partial limit must coincide with $\hat{f}(\cdot,\omega)$ in $\mathcal{O}(U_0)$, which is a contradiction. Therefore, $\hat{f}\in C(\hat{\Pi}_{\mathfrak{a}}(Z_0,\hat{K}_{\mathfrak{a}}))$, and hence $\hat{f}\in\mathcal{O}(\hat{\Pi}_{\mathfrak{a}}(Z_0,\hat{K}_{\mathfrak{a}}))$. □

Lemma 5.4.2. *We have $\cup_{i=1}^m\hat{K}_{\mathfrak{a}}\cdot g_i=\hat{G}_{\mathfrak{a}}$.*

Proof of Lemma. Assuming the opposite, we obtain that there exists $\omega_0\in\hat{G}_{\mathfrak{a}}\setminus\cup_{i=1}^m\hat{K}_{\mathfrak{a}}\cdot g_i$. Since $j_{\mathfrak{a}}(G)$ is dense in $\hat{G}_{\mathfrak{a}}$ (cf. Section 4.1), there exists a sequence $\{h_l\}\subset G$ such that $j_{\mathfrak{a}}(h_l)\to\omega_0$ as $l\to\infty$. Then there exists $1\le i_0\le m$ such that $\{j(h_l)\}\cap\hat{K}\cdot g_{i_0}$ has infinitely many elements. Since $\hat{K}\cdot g_{i_0}$ is closed and $\hat{G}_{\mathfrak{a}}$ is compact, $\hat{K}\cdot g_{i_0}$ is compact as well. Therefore, $\omega_0\in\hat{K}\cdot g_{i_0}$, which contradicts to the assumption that $\cup_{i=1}^m\hat{K}_{\mathfrak{a}}\cdot g_i\subsetneq\hat{G}_{\mathfrak{a}}$. □

Lemma 5.4.3. *There exists a sequence of open sets $\Pi(U_{0,l},K_l)\subset X$, $1\le l\le s$, such that*

(1) $\Pi(U_{0,1},K_1)=\Pi(U_0,K)$, $\Pi(U_{0,s},K_s)=\Pi(U_0,K\cdot g_i)$,

(2) each $U_{0,l}\subset X_0$ is connected, $U_{0,l}\cap U_{0,l+1}\neq\varnothing$, and

(3) $\Pi(U_{0,l}\cap U_{0,l+1},K_l)=\Pi(U_{0,l}\cap U_{0,l+1},K_{l+1})$, $1\le l\le s-1$.

Proof of Lemma. Since covering $p:X\to X_0$ is regular, for each $1\le i\le m$ there exists $h_i\in\pi_1(X_0)$ such that $h_i\cdot\Pi(U_0,K)=\Pi(U_0,K\cdot g_i)$ (i.e., if $x_0\in U_0$ is the base point for $\pi_1(X)$, and we have fixed $y_0\in p^{-1}(x_0)$, then there is a continuous path p^*h_i joining y_0 and $y_0\cdot g_i$). □

Let $\hat{K}_{\mathfrak{a},l}$ denote the closure of K_l in $\hat{G}_{\mathfrak{a}}$. Clearly, the sequence of sets $\hat{\Pi}_{\mathfrak{a}}(U_0,\hat{K}_{\mathfrak{a},l})$, $1\le l\le s$, has the properties analogous to (1) and (3).

We now complete the proof of Theorem 2.16 using an analytic continuation-type argument. Starting with set $\Pi(U_0,K)$ as in the formulation of the theorem, by Lemma 5.4.1 we can extend $f|_{\Pi(U_0,K)}$ to a (unique) function $\hat{f}_1\in\mathcal{O}\big(\hat{\Pi}_{\mathfrak{a}}(U_0,\hat{K}_{\mathfrak{a}})\big)$. Now, let us fix some $1\le i_0\le m$. Let sets $\{U_{0,l}\}$ be as in Lemma 5.4.3. Since $U_0\cap U_{0,2}\ne\varnothing$ is a uniqueness set for holomorphic functions in $\mathcal{O}(U_{0,2})$, we can extend $f|_{\Pi(U_{0,2},K)}$ to a function $\hat{f}_2\in\mathcal{O}\big(\hat{\Pi}_{\mathfrak{a}}(U_{0,2},\hat{K}_{\mathfrak{a},2})\big)$ such that $\hat{f}_1=\hat{f}_2$ on $\hat{\Pi}_{\mathfrak{a}}(U_0\cap U_{0,2},\hat{K}_{\mathfrak{a},2})$. Repeating this argument for the remaining sets $\{U_{0,l}\}$, $3\le l\le s$, we obtain a (unique) extension $\hat{f}_s\in\mathcal{O}\big(\hat{\Pi}_{\mathfrak{a}}(U_{0,2},\hat{K}_{\mathfrak{a}}\cdot g_{i_0})\big)$ of $f|_{\Pi(U_0,K\cdot g_{i_0})}$. The functions $\hat{f}$ and $\hat{f}_s$ coincide on the intersection of their domains $\hat{\Pi}_{\mathfrak{a}}(U_0,\hat{K}_{\mathfrak{a}})\cap\hat{\Pi}_{\mathfrak{a}}(U_0,\hat{K}_{\mathfrak{a}}\cdot g_{i_0})$, since they are continuous and coincide on a dense subset $\iota\big(\Pi(U_0,K)\cap\Pi(U_0,K\cdot g_{i_0})\big)$. Moreover, $\hat{f}_s$ does not depend on the choice of sequence $\{U_{0,l}\}$, as any two paths p^*h_i, p^*h_i' joining y_0 and $y_0\cdot g_i$, cf. Lemma 5.4.3, are homotopic, hence, if $\{U_{0,l}\}$, $\{U_{0,l}'\}$ denote the corresponding sequences of open subsets of X_0, we may assume that $U_{0,l}\cap U_{0,l}'\ne\varnothing$ for all $1\le l\le s$. In turn, each set $U_{0,l}\cap U_{0,l}'$ is a uniqueness sets for functions in $\mathcal{O}(U_{0,l})$, $\mathcal{O}(U_{0,l}')$; from here the uniqueness of continuation follows. Using Lemma 5.4.2, we obtain in this way a (unique) extension of $f|_{p^{-1}(U_0)}$ to a function in $\mathcal{O}(\bar{p}^{-1}(U_0))$. Arguing similarly, we extend function f to a function in $\mathcal{O}(c_{\mathfrak{a}}X)$. Now, Proposition 4.6(2) implies that $f\in\mathcal{O}_{\mathfrak{a}}(X)$, as required.

5.4.2. **Proof of Theorem 2.17.** By the result in [Br3] there exists a (unique) function $F\in\mathcal{O}_{\ell^\infty}(D)\cap C_{\ell^\infty}(\bar{D})$ such that $F|_{\partial D}=f$. Let us show that $F\in\mathcal{O}_{\mathfrak{a}}(D)$. An argument similar to the one in the proof of Proposition 4.6(2) implies that there exists an extension $\tilde{F}\in\mathcal{O}(c_{\ell^\infty}D)\cap C(c_{\ell^\infty}\bar{D})$ of function F from $\bar{D}$ to the fibrewise compactification $c_{\ell^\infty}\bar{D}\subset c_{\ell^\infty}X$ of covering $p|_{\bar{D}}:\bar{D}\to\bar{D}_0$. Let $U_0\subset D_0$ be an open simply connected subset such that $\bar{U}_0\cap\partial D_0\ne\varnothing$ is open in ∂D_0. Below we identify $\Pi(U_0,G)$ with $U_0\times G$, $\hat{\Pi}_{\mathfrak{a}}(U_0,\hat{G}_{\mathfrak{a}})$ with $U_0\times\hat{G}_{\mathfrak{a}}$, and $\hat{\Pi}_{\ell^\infty}(U_0,\hat{G}_{\ell^\infty})$ with $U_0\times\hat{G}_{\ell^\infty}$ (cf. Section 4.5). Define $\hat{F}_{U_0\times\hat{G}_{\mathfrak{a}}}:=(\mathrm{Id}\times\lambda)^*(\tilde{F}|_{U_0\times\hat{G}_{\ell^\infty}})$. We have $F|_{U_0\times G}=(\mathrm{Id}\times j_{\mathfrak{a}})^*\hat{F}_{U_0\times\hat{G}_{\mathfrak{a}}}$ (as a similar relation holds for $\tilde{F}$). Hence, if we can show that $\hat{F}_{U_0\times\hat{G}_{\mathfrak{a}}}\in\mathcal{O}(U_0\times\hat{G}_{\mathfrak{a}})$, then $F|_{U_0\times G}\in\mathcal{O}_{\mathfrak{a}}(U_0\times G)$. Since $U_0\subset D_0$ was chosen arbitrarily, this would imply that $F\in\mathcal{O}_{\mathfrak{a}}(D)$, as needed.

Indeed, by definition $\hat{F}_{U_0\times\hat{G}_{\mathfrak{a}}}(\cdot,\omega)\in\mathcal{O}(U_0)\cap C(\bar{U}_0)$ for all $\omega\in\hat{G}_{\mathfrak{a}}$, hence we only need to show that $\hat{F}_{U_0\times\hat{G}_{\mathfrak{a}}}$ is continuous (cf. Section 4.4). Suppose the opposite, i.e., that there exists a net $\{\omega_\alpha\}\subset\hat{G}_{\mathfrak{a}}$, $\omega_\alpha\to\omega\in\hat{G}_{\mathfrak{a}}$, such that $\hat{F}_{U_0\times\hat{G}_{\mathfrak{a}}}(\cdot,\omega_\alpha)\not\to\hat{F}_{U_0\times\hat{G}_{\mathfrak{a}}}(\cdot,\omega)$ in $\mathcal{O}(U_0)$. By Montel theorem there is a partial limit of $\{\hat{F}_{U_0\times\hat{G}_{\mathfrak{a}}}(\cdot,\omega_\alpha)\}$ in $\mathcal{O}(U_0)$ that does not coincide with $\hat{F}_{U_0\times\hat{G}_{\mathfrak{a}}}(\cdot,\omega)$. Since the restriction $\hat{F}_{U_0\times\hat{G}_{\mathfrak{a}}}|_{\bar{U}_0\cap\partial D_0\times\hat{G}_{\mathfrak{a}}}$ is continuous, and $\bar{U}_0\cap\partial D_0$ is a uniqueness set for functions in $\mathcal{O}(U_0)\cap C(\bar{U}_0)$ (see, e.g., [Bog]), this partial limit must coincide with $\hat{F}_{U_0\times\hat{G}_{\mathfrak{a}}}$ in $\mathcal{O}(U_0)$, a contradiction.

5.5. Proof of Theorems 2.18

5.5.1. **Proof of Theorem 2.18.** We deduce Theorem 2.18 from the following result.

Let D_0 be a relatively compact subdomain of X_0, $D:=p^{-1}(D_0)$. We set $\mathcal{A}_{\mathfrak{a}}(D):=\mathcal{O}_{\mathfrak{a}}(D)\cap C_{\mathfrak{a}}(\bar{D})$ on $D:=p^{-1}(D_0)$ (cf. Introduction for notation). Analogously, we denote by $\mathcal{A}_\iota(D)$ the space of holomorphic functions $f\in\mathcal{A}_{\mathfrak{a}}(D)$ such that for every $x_0\in\bar{D}_0$ the function $g\mapsto f(g\cdot x)$ ($g\in G$, $x\in F_{x_0}$) is in $\mathfrak{a}_\iota$, and by $\mathcal{A}_0(D)$ the $\mathbb{C}$-linear hull of spaces $\mathcal{A}_\iota(D)$, $\iota\in I$.

Theorem 5.5.1. *If X_0 is a Stein manfiold, and $D_0 \subset X_0$ is a strictly pseudoconvex domain, then $\mathcal{A}_0(D)$ is dense $\mathcal{A}_{\mathfrak{a}}(D)$.*

We fix the following

Notation. We denote by $C_{\mathfrak{a}_\iota}X_0$ ($\iota \in I$) the holomorphic Banach vector bundle associated to the principal fibre bundle $p : X \to X_0$ and having fibre $\mathfrak{a}_\iota$ (cf. (2.6)). For a given open subset $D_0 \subset X_0$ we denote by $\mathcal{O}(D_0, C_{\mathfrak{a}_\iota}X_0)$ the space of holomorphic sections of bundle $C_{\mathfrak{a}_\iota}X_0$ over D_0, endowed with the topology of uniform convergence on compact subsets of D_0, which makes it a Fréchet space. We have an isomorphism of Fréchet spaces

$$\mathcal{O}_{\mathfrak{a}_\iota}(D) \overset{\cong}{\to} \mathcal{O}(D_0, C_{\mathfrak{a}_\iota}X_0), \tag{5.21}$$

where $D := p^{-1}(D_0)$ (the proof repeats literally the proof of Proposition 2.11).

Proposition 5.5.2. *Let $Y_0 \Subset X_0$ be open and such that $\bar{Y}_0$ is a holomorphically convex compact subset of X_0, let $D_0 \subset X_0$ be an open neighbourhood of $\bar{Y}_0$. We set $Y := p^{-1}(Y_0)$ and $D := p^{-1}(D_0)$. Also, let X_0 be a Stein manifold, and $f \in \mathcal{O}_{\mathfrak{a}_\iota}(D)$.*

Then for any $\varepsilon > 0$ there exists a function $h \in \mathcal{O}_{\mathfrak{a}_\iota}(X)$ such that $\sup_{z\in Y} |f(z) - h(z)| < C\varepsilon$ for some $C > 0$ independent of $\bar{f}$ and $\varepsilon > 0$.

Proof. We will need the following approximation result due to [Bu2, Theorem C]. Let B be a complex Banach space, let $\mathcal{O}(X_0, B)$ be the space of B-valued holomorphic functions on X_0.

Lemma 5.5.3. *Suppose that $\bar{f} \in \mathcal{O}(D_0, B)$. Then for any $\varepsilon > 0$ there exists a function $\bar{h} \in \mathcal{O}(X_0, B)$ such that $\sup_{z\in Y_0} \|\bar{f}(z) - \bar{h}(z)\|_B < \varepsilon$ for some $C > 0$ independent of $\bar{f}$ and $\varepsilon > 0$.*

Since X_0 is a Stein manifold, there exist holomorphic Banach vector bundles $p_1 : E_1 \to X_0$ and $p_2 : E_2 \to X_0$ with fibres B_1 and B_2, respectively, such that $E_2 = E_1 \oplus C_{\mathfrak{a}_\iota}X_0$ (the Whitney sum) and E_2 is holomorphically trivial, i.e., $E_2 \cong X_0 \times B_2$ (see, e.g. [ZK]). Thus, any holomorphic section of E_2 can be naturally identified with a B_2-valued holomorphic function on X_0. By $q : E_2 \to C_{\mathfrak{a}_\iota}X_0$ and $i : C_{\mathfrak{a}_\iota}X_0 \to E_2$ we denote the corresponding quotient and embedding homomorphisms of these bundles, so that $q \circ i = \mathrm{Id}$. Now, given a function $f \in \mathcal{O}_{\mathfrak{a}_\iota}(D)$, by $\bar{f} \in \mathcal{O}(D_0, C_{\mathfrak{a}_\iota}X_0)$ we denote its image under isomorphism (5.21). Set $\tilde{f} := i(\bar{f}) \in \mathcal{O}(D_0, B_2)$. By Lemma 5.5.3 for every $\varepsilon > 0$ there exists a function $\tilde{h} \in \mathcal{O}(X_0, B_2)$ such that $\sup_{z\in Y_0} \|\tilde{f}(z) - \tilde{g}(z)\|_{B_2} < \varepsilon$. We define $\bar{h} := q(\tilde{h}) \in \mathcal{O}(X_0, C_{\mathfrak{a}_\iota}X_0)$, and denote by $h \in \mathcal{O}_{\mathfrak{a}_\iota}(X)$ the image of $\bar{h}$ under the inverse isomorphism (5.21). The continuity of i and q, and the compactness of $\bar{Y}_0$ now imply that $\sup_{z\in Y} |f(z) - h(z)| < C\varepsilon$ for some $C > 0$ independent of $\bar{f}$ and $\varepsilon > 0$. □

Using this proposition, we now complete the proof of Theorem 2.18. Let $f \in \mathcal{O}_{\mathfrak{a}}(X)$. We have to show that for some open subsets $Y_{0,k} \Subset Y_{0,k+1} \Subset X_0$, $k \geq 1$, for any $\varepsilon > 0$ there exist functions $h_k \in \mathcal{O}_{\mathfrak{a}_0}(X)$ such that $\sup_{x\in Y_k} |f(x) - h_k(x)| < \frac{\varepsilon}{k}$, where $Y_k := p^{-1}(Y_{0,k})$. Since X_0 is a Stein manifold, we may assume without loss of generality that each $\bar{Y}_{0,k}$, $k \geq 1$, is holomorphically convex in X_0. There is a strictly pseudoconvex open neighbourhood $D_{0,k} \Subset X_0$ of $\bar{Y}_{0,k}$, $k \geq 1$ (see, e.g., [HL]). Since the restriction $f|_{\bar{D}_k} \in \mathcal{A}_{\mathfrak{a}}(D_k)$, where $D_k := p^{-1}(D_{0,k})$, by Theorem 5.5.1 there exist functions $h'_k \in \mathcal{A}_{\mathfrak{a}_0}(D_k)$, $k \geq 1$, such that $\sup_{x\in D_k} |f(x) - h'_k(x)| < \frac{\varepsilon}{2k}$. By the definition of space $\mathcal{A}_{\mathfrak{a}_0}(D_k)$, there exists $\iota_k \in I$ such that $h'_k \in \mathcal{A}_{\iota_k}(D_k)$, $k \geq 1$. Now, by Proposition 5.5.2 there exists a function $h_k \in \mathcal{O}_{\mathfrak{a}_{\iota_k}}(X)$ such that $\sup_{x\in Y_k} |h'_k(x) - h_k(x)| < \frac{\varepsilon}{2k}$. Therefore, $\sup_{x\in Y_k} |f(x) - h_k(x)| < \frac{\varepsilon}{2k}$. Since $\mathcal{O}_{\mathfrak{a}_{\iota_k}}(X) \subset \mathcal{O}_{\mathfrak{a}_0}(X)$, this implies the required, modulo Theorem 5.5.1.

5.5.1.1. *Proof of Theorem 5.5.1. Notation.* We denote by $\mathcal{A}(D_0, C_{\mathfrak{a}}X_0)$ and $\mathcal{A}(D_0, C_{\mathfrak{a}_\iota}X_0)$ the spaces of sections of bundles $C_{\mathfrak{a}}X_0$ and $C_{\mathfrak{a}_\iota}X_0$, respectively, continuous over $\bar{D}_0$ and holomorphic on D_0. Space $\mathcal{A}(D_0, C_{\mathfrak{a}}X_0)$ is endowed with norm $\|f\| := \sup_{x\in\bar{D}_0}\|f(x)\|_{\mathfrak{a}}$, which makes it a Banach space. Then $\mathcal{A}(D_0, C_{\mathfrak{a}_\iota}X_0)$ is a closed subspace of $\mathcal{A}(D_0, C_{\mathfrak{a}}X_0)$. We also define linear space $\mathcal{A}_0(D_0, C_{\mathfrak{a}}X_0) := \bigcup_{\iota\in I}\mathcal{A}(D_0, C_{\mathfrak{a}_\iota}X_0)$. We have the following isomorphisms of Banach spaces:

$$\mathcal{A}_{\mathfrak{a}_\iota}(D) \overset{\cong}{\to} \mathcal{A}(D_0, C_{\mathfrak{a}_\iota}X_0), \quad \mathcal{A}_0(D) \overset{\cong}{\to} \mathcal{A}_0(D_0, C_{\mathfrak{a}}X_0) \tag{5.22}$$

(the proof is similar to the proof of Proposition 2.11). In view of (5.22), Theorem 5.5.1 can be restated as follows:

($*$) *Suppose that X_0 is a Stein manifold, and the subdomain $D_0 \Subset X_0$ is strictly pseudoconvex. Then $\mathcal{A}_0(D_0, C_{\mathfrak{a}}X_0)$ is dense $\mathcal{A}(D_0, C_{\mathfrak{a}}X_0)$.*

For the proof of this claim, we will need the following notation and results. Let E be a holomorphic Banach vector bundle on D_0. Similarly, we denote by $\Lambda_b^{(0,q)}(D_0, E)$, $q \ge 0$, the Banach space of bounded continuous $(0,q)$-forms on D_0 with values in the bundle E, endowed with sup-norm defined analogously to (1.16). Using Lemma 5.1.1 and the result in [ZK] (cf. a remark after Theorem 2.9), we obtain the following

Corollary 5.5.4. *There exists a bounded linear operator*

$$R_{D_0,E} \in \mathcal{L}\left(\Lambda_b^{(0,q)}(D_0, E), \Lambda_b^{(0,q-1)}(D_0, E)\right), \quad q \ge 1,$$

such that if $\omega \in \Lambda_b^{(0,q)}(D_0, E)$ is C^∞ and satisfies $\bar\partial\omega = 0$ on D_0, then $\bar\partial R_{D_0,E}\omega = \omega$ on D_0.

We define $\mathcal{A}(D_0, B) := \mathcal{O}(D_0, B)\cap C(\bar{D}_0, B)$, and endow space $\mathcal{A}(D_0, B)$ with norm $\|f\|_{\bar{D}_0} := \sup_{x\in\bar{D}_0}\|f(x)\|_B$. The next result follows easily from the results in [HL] (proved for the scalar case $B = \mathbb{C}$), since all integral presentations and estimates are preserved when passing to Banach-valued forms.

Lemma 5.5.5. *Let $K \subset \mathcal{A}(D_0, B)$ be compact. Then for every $\varepsilon > 0$ there exists an open neighbourhood $D_0' \subset X_0$ of $\bar{D}_0$ and an operator $A_{K,\varepsilon} = A_{D_0,K,\varepsilon} \in \mathcal{L}\big(\mathcal{A}(D_0, B), \mathcal{A}(D_0', B)\big)$ such that for each $f \in K$ we have $\|f - Af|_{\bar{D}_0}\|_{\bar{D}_0} < \varepsilon$.*

We prove claim ($*$) in three steps (i)-(iii).

(i) Let $f \in \mathcal{A}(D_0, C_{\mathfrak{a}}X_0)$. Combining the result in [ZK] (cf. a remark after Theorem 2.9) and Lemma 5.5.5, we may assume that $f \in \mathcal{O}(D_0', C_{\mathfrak{a}}X_0)$, where $D_0' \subset X_0$ is an open neighbourhood of $\bar{D}_0$.

We have to show that for every $\varepsilon > 0$ there exists a section $F \in \mathcal{A}_0(D_0, C_{\mathfrak{a}}X_0)$ such that $\sup_{x\in\bar{D}_0}\|f(x) - F(x)\|_{\mathfrak{a}} < \varepsilon$.

(ii) Let $\mathcal{U} = \{U_k\}_{k=1}^M$, where each $U_k \Subset D_0'$ is open and biholomorphic to an open polydisk in $\mathbb{C}^n$, and $D_0 \Subset \cup_{k=1}^M U_k$.

Lemma 5.5.6. *For every $\varepsilon > 0$ there exist a subspace $\mathfrak{a}_{\iota_\varepsilon} \subset \mathfrak{a}$ ($\iota_\varepsilon \in I$) and sections $F_{\varepsilon,k} \in \mathcal{A}\big(U_k, C_{\mathfrak{a}_{\iota_\varepsilon}}X_0\big)$ such that*

$$\|f(x) - F_{\varepsilon,k}(x)\|_{\mathfrak{a}} < \varepsilon \quad \textit{for all } x \in U_k, \quad 1 \le k \le M. \tag{5.23}$$

Proof. Since each U_k, $1 \le k \le M$, is simply connected, bundles $C_{\mathfrak{a}}X_0$, $C_{\mathfrak{a}_\iota}X_0$ ($\iota \in I$) admit holomorphic trivializations over U_k. Throughout the proof, we identify sections of these bundles over U_k with corresponding $\mathfrak{a}$-valued (respectively, $\mathfrak{a}_\iota$-valued) functions on U_0.

By our assumption, for every $1 \le k \le M$ there exists a biholomorphism ψ_k between U_k and open polydisk $\Delta \subset \mathbb{C}^n$ centered at 0. Without loss of generality, we may assume that $f|_{U_k}$ is defined over an open neighbourhood of $\bar{\Delta}$. We denote $f|_{U_k}$ by f_k, so that f_k can be identified by means of the corresponding holomorphic trivialization of bundle $C_{\mathfrak{a}}X_0$ with a holomorphic $\mathfrak{a}$-valued function defined on an open neighbourhood of $\bar{\Delta}$.

For a given function $h \in \mathcal{O}\big(\Delta, \mathfrak{a}\big)$ we denote by $T_0^N h$ its Taylor polynomial at $x = 0$ of order N. Without loss of generality we may assume that N is chosen in such a way that

$$\|f_k(x) - T_0^N f_k(z)\|_{\mathfrak{a}} < \frac{\varepsilon}{2}, \quad \text{for all } x \in \Delta, \quad 1 \le k \le M,$$

where $T_0^N f_k(x) := \sum_{|\alpha| \le N} a_{k,\alpha}x^\alpha$, $a_{k,\alpha} \in \mathfrak{a}$, and α is a multi-index. Since $\mathfrak{a}_0$ is dense in $\mathfrak{a}$, for every $\delta > 0$ and all $1 \le k \le M$, $|\alpha| \le N$, there exist $a^\varepsilon_{k,\alpha} \in \mathfrak{a}_0$ such that $\|a_{k,\alpha} - a^\varepsilon_{k,\alpha}\|_{\mathfrak{a}} < \delta$. We choose $\delta > 0$ to be sufficiently small, so that

$$\left\| \sum_{|\alpha| \le N} a_{k,\alpha}x^\alpha - F_{\varepsilon,k}(x) \right\|_{\mathfrak{a}} < \frac{\varepsilon}{2},$$

where $F_{\varepsilon,k}(x) := \sum_{|\alpha| \le N} a^\varepsilon_{k,\alpha}x^\alpha$. Therefore,

$$\|f_k(x) - F_{\varepsilon,k}(x)\|_{\mathfrak{a}} < \varepsilon, \quad \text{for all } x \in \Delta, \quad 1 \le k \le M.$$

By definition, there exists $\iota_\varepsilon \in I$ such that $\mathfrak{a}_{\iota_\varepsilon}$ contains all $a^\varepsilon_{k,\alpha}$ ($1 \le k \le M$, $|\alpha| \le N$); hence $F_{\varepsilon,k}(x) \in \mathcal{A}\big(\Delta, \mathfrak{a}_{\iota_\varepsilon}\big)$. $\square$

(iii). We will also need the following result.

Lemma 5.5.7. *In the notation of Lemma 5.5.6, for every $\varepsilon > 0$ there exists a section $F \in \mathcal{A}(D_0, C_{\mathfrak{a}_{\iota_\varepsilon}}X_0) \subset \mathcal{A}_0(D_0, C_{\mathfrak{a}}X_0)$ such that*

$$\|F(x) - F_{\varepsilon,k}(x)\|_{\mathfrak{a}} < C\varepsilon, \quad \text{for all } x \in U_k \cap \bar{D}_0, \quad 1 \le k \le M$$

for a certain $C > 0$ independent of the section $f \in \mathcal{A}_0(D_0', C_{\mathfrak{a}}X_0)$ and $\varepsilon > 0$.

Proof. There exists an open neighbourhood $D_0'' \Subset D_0'$ of $\bar{D}_0$ such that $D_0'' \Subset \cup_{k=1}^M U_k$. We may assume without loss of generality that D_0'' is strictly pseudoconvex.

Let $\{\rho_k\} \subset C^\infty(X_0)$ be a collection of functions such that $\operatorname{supp}(\rho_k) \Subset U_k$, $1 \le k \le M$, and $\sum_{k=1}^m \rho_k \equiv 1$ on $\bar{D}_0''$. Let $c := \max_{x \in \bar{D}_0''} \sum_{k=1}^M |\bar{\partial}\rho_k(x)| < \infty$ (in some local coordinates on D_0').

We define holomorphic 1-cocycle: if $U_k \cap U_l \ne \varnothing$ then

$$g_{k,l} := F_{\varepsilon,k}|_{U_k \cap U_l \cap D_0''} - F_{\varepsilon,l}|_{U_k \cap U_l \cap D_0''} \in \mathcal{A}(U_k \cap U_l \cap D_0'', C_{\mathfrak{a}_{\iota_\varepsilon}}X_0),$$

and $g_{k,l} := 0$ if $U_k \cap U_l \cap D_0'' = \varnothing$. According to Lemma 5.5.6

$$\sup_{x \in U_k \cap U_l \cap D_0''} \|g_{k,l}\|_{\mathfrak{a}} < 2\varepsilon.$$

Here we assume without loss of generality that $\{U_k \cap D_0''\}_{k=1}^M$ are the coordinate charts in the definition of norm in $\Lambda_b^{0,1}(D_0'', C_{\mathfrak{a}_{\iota_\varepsilon}}X_0)$, cf. (1.16).

We resolve cocycle $\{g_{\varepsilon,k,l}\}$ by the formula $\tilde{g}_l := \sum_{k=1}^M \rho_k g_{k,l} \in C^\infty(U_l \cap D_0'')$, so that $g_{k,l} = \tilde{g}_k - \tilde{g}_l$ on $U_k \cap U_l \cap D_0''$. It follows that the $(0,1)$-form ω defined by the formula $\omega := \bar{\partial}\tilde{g}_l$ on $U_l \cap D_0''$ is a $\bar{\partial}$-closed form on D_0'' taking values in bundle $C_{\mathfrak{a}_{\iota_\varepsilon}}X_0$.

We have $\|\omega\|_{D_0''} \le 2c\varepsilon$. By Corollary 5.5.4 there exists a function $\eta \in \Lambda_b^{0,0}(D_0'', E)$ such that $\bar\partial\eta = \omega$ and $\|\eta\|_{D_0''} \le C_1\|\omega\|_{D_0''} \le 2C_1c\varepsilon$ for some constant $C_1 > 0$ independent of ω.

Since $D_0 \Subset D_0''$, the restriction $\eta|_{\bar D_0}$ is continuous on $\bar D_0$. We define

$$F := F_{\varepsilon,k}|_{U_k\cap\bar D_0} - \tilde g_k|_{U_k\cap\bar D_0} + \eta|_{U_k\cap\bar D_0}, \quad 1 \le k \le M.$$

It follows that $F \in \mathcal{A}(D_0, C_{\mathfrak{a}_{\iota_\varepsilon}}X_0)$ and

$$\sup_{x\in\bar D_0} \|F - F_{\varepsilon,k}\|_{\mathfrak{a}} \le 2M\varepsilon + 2C_1c\varepsilon,$$

so we may set $C := 2M + 2C_1c$. □

The proof of claim (∗) now follows from Lemmas 5.5.6 and 5.5.7.

5.6. Proof of Theorem 2.9

We will need the following

Definition 5.6.1. A closed subset $Y \subset c_{\mathfrak{a}}X$ is called a complex submanifold of codimension k if for every $x \in c_{\mathfrak{a}}X$ there exist a neighbourhood $U = \hat\Pi(U_0, K)$ of x, where $U_0 \subset X_0$ is open and simply connected, $K \subset \hat G_{\mathfrak{a}}$ is open (cf. (4.14)), and functions $h_1, \dots, h_k \in \mathcal{O}(U)$ such that
(1) $Y \cap U = \{x \in U : h_1(x) = \cdots = h_k(x) = 0\}$;
(2) The rank of map $z \mapsto \big(h_1(z,\omega), \dots, h_k(z,\omega)\big)$ is k at each point $(z,\omega) \in Y \cap U$.

Any complex submanifold Y of $c_{\mathfrak{a}}X$ has the following properties:
(i) $\iota^{-1}\big(Y\big) \subset X$ is a complex submanifold of X of codimension k.
(ii) $Y \cap \iota(X)$ is dense in Y (cf. Lemma 5.6.5 below).

Suppose that $\mathfrak{a}$ is self-adjoint. It is immediate that if $Z \subset X$ is a complex $\mathfrak{a}$-submanifold (cf. Definition 2.6), then the closure of $\iota(Z)$ in $c_{\mathfrak{a}}X$ is a complex submanifold of $c_{\mathfrak{a}}X$; conversely, if Y is a complex submanifold of $c_{\mathfrak{a}}X$, then $\iota^{-1}(Y) \subset X$ is a complex $\mathfrak{a}$-submanifold.

Definition 5.6.2. A function $f \in C(Y)$ is called holomorphic if $\iota^* f \in \mathcal{O}\big(\iota^{-1}(Y)\big)$.
The algebra of holomorphic functions on Y is denoted by $\mathcal{O}(Y)$.

Proposition 5.6.3. *Suppose that $\mathfrak{a}$ is self-adjoint, Y is a complex submanifold of $c_{\mathfrak{a}}X$, denote $Z := \iota^{-1}(Y)$. Then $\mathcal{O}_{\mathfrak{a}}(Z) \cong \iota^*\mathcal{O}(Y)$ (so every function in $f \in \mathcal{O}_{\mathfrak{a}}(Z)$, cf. Definition 2.8, admits a unique extension to a function $\hat f \in \mathcal{O}(Y)$ such that $f = \iota^*\hat f$).*

Theorem 5.6.4. *Let X_0 is a Stein manifold, $Y \subset c_{\mathfrak{a}}X$ be a complex submanifold, $f \in \mathcal{O}(Y)$. Then there exists a function $F \in \mathcal{O}(c_{\mathfrak{a}}X)$ such that $F|_Y = f$.*

In view of Propositions 4.6, 5.6.3 and the remark before Definition 5.6.2, Theorem 2.9 is a special case of Theorem 5.6.4.

5.6.1. **Proof of Proposition 5.6.3.** First, let f be a holomorphic $\mathfrak{a}$-function on Z in the sense of Definition 2.8, i.e., there is a function $F \in C_{\mathfrak{a}}(X)$ such that $F|_Z = f$. By Proposition 4.6(1) there exists a function $\hat F \in C(c_{\mathfrak{a}}X)$, such that $\iota^*\hat F = F$. We set $\hat f := \hat F|_Y$. Since $\iota^*\hat f = f$, we obtain $\hat f \in \mathcal{O}(Y)$ (cf. Definition 5.6.2), as required.

Now, let $\hat f \in \mathcal{O}(Y)$. Since $c_{\mathfrak{a}}X$ is a normal space, by Tietze-Urysohn extension theorem there exists a function $\hat F \in C(c_{\mathfrak{a}}X)$ such that $\hat F|_Y = \hat f$. By Proposition 4.6(2) $F := \iota^*\hat F$ belongs to $C_{\mathfrak{a}}(X)$. Since $F|_Z = f$, function f $(= \iota^*\hat f)$ is a holomorphic $\mathfrak{a}$-function on Z in the sense of Definition 2.8.

5.6.2. **Proof of Theorem 5.6.4.** Let Y be a complex submanifold of $c_{\mathfrak{a}}X$, $y_0 \in Y_0$. The following lemma is a straightforward consequence of the inverse function theorem.

Lemma 5.6.5. *There exist an open neighbourhood $V \subset c_{\mathfrak{a}}X$ of y_0, open subsets $V_0 \subset X_0$ and $L \subset \hat{G}_{\mathfrak{a}}$, a closed complex submanifold Z_0 of V_0 (of the same codimension as Y), and a biholomorphic map $\Phi \in \mathcal{O}(V_0 \times L, V)$ such that $\Phi^{-1}(V \cap Y) = Z_0 \times L$.*

The next lemma follows easily from Lemma 5.6.5.

Lemma 5.6.6. *Let $f \in \mathcal{O}(Y)$. For every $y \in Y$ there exist a neighbourhood $U \subset c_{\mathfrak{a}}X$ of y and a function $F_U \in \mathcal{O}(U)$ such that $F_U|_{U\cap Y} = f|_{U\cap Y}$.*

Lemma 5.6.7. *The sheaf of ideals I_Y is coherent.*

Proof. By Lemma 5.6.5, it suffices to prove the coherence of sheaf $I_Z \subset \mathcal{O}_V$, where $Z := Z_0 \times L$, $V := V_0 \times L$, $Z_0 \subset V_0$ (we use notation of Lemma 5.6.5), $\mathcal{O}_V =: \mathcal{O}$ is the structure sheaf on V. We denote by $\tilde{\mathcal{O}}$ the structure sheaf on V_0, and by $\tilde{I}_{Z_0} \subset \tilde{\mathcal{O}}$ the sheaf of ideals of Z_0. By Cartan theorem (see, e.g., [Gun3]) $\tilde{I}_{Z_0}$ is a $\tilde{\mathcal{O}}$-coherent sheaf, i.e. for every point in V_0 there is an open neighbourhood (without loss of generality, we may assume that it coincides with V_0) over which there exists a free resolution

$$0 \longrightarrow \tilde{\mathcal{O}}^{m_N} \xrightarrow{\tilde{\varphi}_{N-1}} \dots \xrightarrow{\tilde{\varphi}_1} \tilde{\mathcal{O}}^{m_1} \dots \xrightarrow{\tilde{\varphi}_0} \tilde{I}_{Z_0} \longrightarrow 0, \tag{6.24}$$

where sheaf homomorphisms $\tilde{\varphi}_k$ are given by functions in $\mathcal{O}\big(V_0, M_{m_k\times m_{k+1}}(\mathbb{C})\big)$, $0 \le k \le N-1$ (here $m_0 := 1$); (6.24) can be rewritten as a collection of short exact sequences

$$0 \longrightarrow \tilde{\mathcal{R}}_k \xrightarrow{\iota} \tilde{\mathcal{O}}^{m_k} \xrightarrow{\tilde{\varphi}_{k-1}} \tilde{\mathcal{R}}_{k-1} \longrightarrow 0, \quad 1 \le k \le N-1, \tag{6.25}$$

where $\tilde{\mathcal{R}}_k := \mathrm{Im}\ \tilde{\varphi}_k$ $(1 \le k \le N-1)$, $\tilde{\mathcal{R}}_0 := \tilde{I}_{Z_0}$, and ι stands for inclusion.

Let $y_0 = (x_0, \omega_0) \in V := V_0 \times L$, $f_{y_0} \in \mathcal{O}(U)$ be a section of sheaf $\mathcal{O}^{m_k}$ over an open neighbourhood $U := U_0 \times K \subset V$ of y_0. Let map $(\varphi_k)_*(f_{y_0})$ $(0 \le k \le N-1)$ be defined by

$$U \ni (x,\omega) \mapsto \big((\tilde{\varphi}_k)_*(f_{y_0}(\cdot,\omega)\big),$$

where $(\tilde{\varphi}_k)_*$ is the homomorphism of sections over U_0 induced by $\tilde{\varphi}_k$. This is a section over U of sheaf $\mathcal{O}^{m_{k-1}}$ if $k \ge 1$, or sheaf I_Z if $k = 0$. The homomorphisms of sections $(\varphi_k)_*$ induce sheaf homomorphisms φ_k, $0 \le k \le N-1$. Thus, we get a chain complex

$$0 \longrightarrow \mathcal{O}^{m_N} \xrightarrow{\varphi_{N-1}} \dots \xrightarrow{\varphi_1} \mathcal{O}^{m_1} \dots \xrightarrow{\varphi_0} I_Z \longrightarrow 0, \tag{6.26}$$

or, equivalently, a collection of chain complexes

$$0 \longrightarrow \mathcal{R}_k \xrightarrow{\iota} \mathcal{O}^{m_k} \xrightarrow{\varphi_{k-1}} \mathcal{R}_{k-1} \longrightarrow 0, \quad 1 \le k \le N-1, \tag{6.27}$$

where $\mathcal{R}_k := \mathrm{Im}\ \varphi_k$ $(1 \le k \le N-1)$, $\mathcal{R}_0 := I_Z$, and ι stands for inclusion. We have to show that sequence (6.26) is exact: this would imply that I_Z is coherent. The latter is equivalent to exactness of sequences (6.27).

Since ι is an inclusion, it is injective and its image coincides with the kernel of φ_{k-1}. Therefore, we only need to show that homomorphisms φ_{k-1}, $1 \le k \le N-1$, in (6.27) are surjective.

We fix $1 \le k \le N-1$. let $g_{y_0} \in \Gamma(U, \mathcal{R}_{k-1})$ be a section of sheaf $\mathcal{R}_{k-1}$ over U. By shrinking K we may assume that g_{y_0} is defined also over $U_0 \times \bar{K}$, and that $\bar{K} \subset L$ ($\bar{K}$ is compact as $\hat{G}_{\mathfrak{a}}$ is compact). To show that φ_{k-1} is surjective, it suffices to prove that there exists a section $f_{y_0} \in \Gamma(U, \mathcal{O}^{m_k})$ such that $(\varphi_{k-1})_*(f_{y_0}) = g_{y_0}$, where $(\varphi_{k-1})_*$ is the homomorphism of sections induced by φ_{k-1}. We may assume, without loss of generality, that $U_0 \subset V_0$ are polydisks in

$\mathbb{C}^n$. Let $\Gamma(U_0, \tilde{\mathcal{R}}_{k-1})$ and $\Gamma(U_0, \tilde{\mathcal{O}}^{m_k})$ denote the Fréchet spaces of sections over U_0 of sheaves $\tilde{\mathcal{R}}_{k-1}$ and $\tilde{\mathcal{O}}^{m_k}$, respectively, endowed with the topology of uniform convergence on compact subsets of U_0. Since (6.25) is exact, the sequence of sections

$$0 \longrightarrow \Gamma(U_0, \tilde{\mathcal{R}}_k) \longrightarrow \Gamma(U_0, \tilde{\mathcal{O}}^{m_k}) \longrightarrow \Gamma(U_0, \tilde{\mathcal{R}}_{k-1}) \longrightarrow 0,$$

is also exact (see, e.g., [Gun3]), therefore by open mapping theorem homomorphism $\tilde{\varphi}_{k-1}$ induces an isomorphism of Fréchet spaces

$$\Gamma(U_0, \tilde{\mathcal{O}}^{m_k})/\Gamma(U_0, \tilde{\mathcal{R}}_k) \cong \Gamma(U_0, \tilde{\mathcal{R}}_{k-1}). \tag{6.28}$$

The section g_{y_0} determines a continuous map $\hat{g} \in C\big(\bar{K}, \Gamma(U_0, \tilde{\mathcal{R}}_{k-1})\big)$ defined by $\hat{g}(\omega) := g_{y_0}(\cdot, \omega)$ ($\omega \in \bar{K}$). By Michael selection theorem, in view of (6.28), there exists a continuous selection $\hat{f}$ for $\hat{g}$ in $\Gamma(U_0, \tilde{\mathcal{O}}^{m_k})$, i.e., a map $\hat{f} \in C\big(\bar{K}, \Gamma(U_0, \tilde{\mathcal{O}}^{m_k})\big)$ such that $(\varphi_{k-1})_* \circ \hat{f} = \hat{g}$. We set $f_{y_0}(\cdot, \omega) := \hat{f}(\omega)$, $\omega \in K$. Clearly, $f_{y_0} \in \Gamma(U, \mathcal{O}^{m_k})$. Thus, φ_{k-1} is surjective, as needed □

Proof of Theorem 5.6.4 is standard. Namely, by Lemma 5.6.6 there exist an open cover $\mathcal{U} = \{U_j\}$ of $c_{\mathfrak{a}}X$ and functions $f_j \in \mathcal{O}(U_j)$ such that $f_j|_{Y \cap U_j} = f|_{Y \cap U_j}$ if $Y \cap U_j \neq \varnothing$; if $Y \cap U_j = \varnothing$, we define $f_j := 0$. Then $\{g_{ij} := f_i - f_j$ on $U_i \cap U_j \neq \varnothing\}$ is a 1-cocylce with values in I_Y. By Lemma 5.6.7 sheaf I_Y is coherent, so by Theorem 4.10(B) $H^1(c_{\mathfrak{a}}X, I_Y) = 0$. Therefore, possibly after passing to a refinement of $\mathcal{U}$, we can find holomorphic functions $h_j \in \Gamma(U_j, I_Y)$ such that $g_{ij} = h_i - h_j$ on $U_i \cap U_j \neq \varnothing$. Now, we define F as $F := f_j - h_j$ on U_j, for all j. □

5.7. Proof of Theorem 2.7

Our proof is based on Theorem 4.10(A), and the equivalence of notions of a complex $\mathfrak{a}$-submanifold of X and a complex submanifold of $c_{\mathfrak{a}}X$ established in Section 5.6.

It suffices to prove that, given a complex submanifold $Y \subset c_{\mathfrak{a}}X$ of codimension k, there exists a countable collection of functions $f_i \in \mathcal{O}(c_{\mathfrak{a}}X)$, $i \in I$, such that

(i) $Y = \{y \in c_{\mathfrak{a}}X : f_i(y) = 0$ for all $i \in I\}$, and

(ii) for each $y_0 \in Y$ there exists a neighbourhood $U = \hat{\Pi}(U_0, K)$ (cf. (4.14) for notation) and functions $f_{i_1}, \dots, f_{i_k}$ such that $Y \cap U = \{y \in U : f_{i_1}(y) = \cdots = f_{i_k} = 0\}$, and the rank of map $z \to \big(f_1(z,\omega), \dots, f_k(z,\omega)\big)$, $(z,\omega) \in U$, is maximal at each point of $Y \cap U$.

The sheaf of ideals I_Y of Y is coherent (cf. the proof of Corollary 5.6.7), hence by Theorem 4.10(A) there exists a countable collection of sections $f_i \in \Gamma(c_{\mathfrak{a}}X, I_Y)$ ($\subset \mathcal{O}_{c_{\mathfrak{a}}X}$), $i \in I$, that generate I_Y at every point. By definition, condition (i) is satisfied, and for every point $y_0 \in Y$ there is a neighbourhood $U = \hat{\Pi}(U_0, K)$, sections $f_{i_1}, \dots, f_{i_m}$, and functions $u_{jl} \in \mathcal{O}(c_{\mathfrak{a}}X)$ such that

$$h_j = u_{j1}\tilde{f}_1 + \cdots + u_{jm}\tilde{f}_m, \quad 1 \leq j \leq k, \tag{7.29}$$

where $\tilde{f}_l := f_{i_l}$, $1 \leq l \leq m$, and h_j are the generators of $I_Y|_U$ (such that in appropriate local coordinates $h_j(z,\omega) = z_j$, the j-th component of $z \in U_0 \subset \mathbb{C}^n$, $\omega \in K$, cf. Lemma 5.6.5).

It is immediate that $Y \cap U = \{y \in U : \tilde{f}_1(y) = \cdots = \tilde{f}_m = 0\}$. It remains to show that functions $\tilde{f}_l$ can be chosen in such a way that $m = k$, and condition (ii) is satisfied.

Indeed, let ∇h_j, $\nabla \tilde{f}_l$ denote the vector-valued functions $\nabla_z h_j(z,\omega)$, $\nabla_z \tilde{f}_l(z,\omega)$, $(z,\omega) \in U$. Then

$$\nabla h_j = u_{j1}\nabla\tilde{f}_1 + \cdots + u_{jm}\nabla\tilde{f}_m \quad \text{ on } Y \cap U, \quad 1 \leq j \leq k.$$

Since $(\nabla h_j)_{j=1}^k$ has rank k on U, we obtain that $k \leq m$, and $(u_{jl})_{1 \leq j \leq k, 1 \leq l \leq m}$, $(\nabla \tilde{f}_l)_{l=1}^m$ have rank k. Assuming that vectors $(u_{jl_1})_{j=1}^k, \dots, (u_{jl_k})_{j=1}^k$ and $\nabla\tilde{f}_{l_1}, \dots, \tilde{f}_{l_k}$ are linearly independent, we can apply the holomorphic Inverse function theorem to the matrix identity (7.29) (possibly, after shrinking U and, as a result, extending collection $\{f_i\}_{i\in I}$) to represent functions $\tilde{f}_l$, $l\neq l_i$, $1\le i\le k$ via $\tilde{f}_{l_1},\dots,\tilde{f}_{l_k}$, so that we can choose as $f_{i_1},\dots,f_{i_k}$ the functions $\tilde{f}_{l_1},\dots,\tilde{f}_{l_k}$. This completes the proof.

5.8. Proofs of Theorems 2.14 and 2.15

5.8.1. **Divisors on $c_{\mathfrak{a}}X$.** For the proofs of Theorems 2.14 and 2.15 we will need the following

Definition 5.8.1. An (effective) divisor D on $c_{\mathfrak{a}}X$ is given by an open cover $\{U_\alpha\}$ of $c_{\mathfrak{a}}X$ and functions $f_\alpha\in\mathcal{O}(U_\alpha)$ not identically equal to zero on any open subset of U_α, such that $f_\alpha=d_{\alpha\beta}f_\beta$ on $U_\alpha\cap U_\beta\neq\varnothing$ for a nowhere zero function $d_{\alpha\beta}\in\mathcal{O}(U_\alpha\cap U_\beta)$, for all α,β.

The divisors on $c_{\mathfrak{a}}X$ form a multiplicative semigroup $\mathrm{Div}(c_{\mathfrak{a}}X)$ with identity.

We say that divisors $D=\{(U_\alpha,f_\alpha\}$, $D'=\{(U'_\beta,f'_\beta)\in\mathrm{Div}(c_{\mathfrak{a}}X)$ are equivalent if there exists a refinement $\{V_\gamma\}$ of both covers $\{U_\alpha\}$ and $\{U'_\beta\}$ and nowhere zero holomorphic functions c_γ on V_γ such that $f_\alpha|_{V_\gamma}=c_\gamma f'_\beta|_{V_\gamma}$ for $V_\gamma\subset U_\alpha\cap U'_\beta$.

Similarly, we define principal divisors as those divisors $\mathrm{Div}(c_{\mathfrak{a}}X)$ that are determined by functions in $\mathcal{O}(c_{\mathfrak{a}}X)$.

Suppose that $\mathfrak{a}$ is self-adjoint. The pullback $E=\iota^*D$ of a divisor $D\in\mathrm{Div}(c_{\mathfrak{a}}X)$ is an $\mathfrak{a}$-divisor in the sense of Definition 2.13. Since $\iota(X)$ is dense in $c_{\mathfrak{a}}X$, by Hurwitz theorem divisor D is determined by E uniquely. Conversely, a divisor E on X is an $\mathfrak{a}$-divisor only if $E=\iota^*D$ for some $\mathrm{Div}(c_{\mathfrak{a}}X)$.

It is immediate that if divisors D_1, $D_2\in\mathrm{Div}(c_{\mathfrak{a}}X)$ are equivalent, then $E_1=\iota^*D_1$, $E_2=\iota^*D_2\in\mathrm{Div}_{\mathfrak{a}}(X)$ are equivalent (in $\mathrm{Div}(X)$). Also, a divisor $D\in\mathrm{Div}(c_{\mathfrak{a}}X)$ is principal if and only if $E=\iota^*D\in\mathrm{Div}_{\mathfrak{a}}(X)$ is principal.

Lemma 5.8.2. *Every open cover of $c_{\mathfrak{a}}X$ has an at most countable refinement by open sets $\hat{\Pi}(U_{0,\alpha},K_\alpha)$ (cf. (4.14)), where $U_{0,\alpha}\subset X_0$ is open and simply connected, and $K_\alpha\in\mathfrak{Q}$ (cf. (4.12)).*

Thus, we may assume that in Definition 5.8.1 the open cover $\{U_\alpha\}$ is at most countable, and $U_\alpha=\hat{\Pi}(U_{0,\alpha},K_\alpha)$, for all α.

We prove Theorem 2.15 first.

5.8.2. **Proof of Theorem 2.15.** By the results of Section 5.8.1, it suffices to prove the following: *let X_0 be a Stein manifold, $D\in\mathrm{Div}(c_{\mathfrak{a}}X)$; if X_0 is homotopically equivalent to an open subset $Y_0\subset X_0$ such that the restriction of divisor D to $Y:=\bar{p}^{-1}(Y_0)$ is equivalent to a principal divisor in $\mathrm{Div}(c_{\mathfrak{a}}X)$, then D itself is equivalent to a principal divisor.*

Let $\mathcal{O}_*(U)$, where $U\subset c_{\mathfrak{a}}X$ is open, denote the multiplicative group of nowhere zero holomorphic functions on U. We denote by $\mathcal{O}_*$ the multiplicative sheaf of germs of nowhere zero holomorphic functions on $c_{\mathfrak{a}}X$. In what follows, we use notation of Definition 5.8.1.

The functions $\{d_{\alpha\beta}\in\mathcal{O}_*(U_\alpha\cap U_\beta)\}$ are determined uniquely, and define a 1-cocycle with values in sheaf $\mathcal{O}_*$. It is easy to see that divisor D is equivalent to a principal divisor if and only if there exist functions $d_\alpha\in\mathcal{O}_*(U_\alpha)$ such that

$$d_{\alpha\beta}=d_\beta d_\alpha^{-1}\quad\text{on } U_\alpha\cap U_\beta\neq\varnothing, \tag{8.30}$$

for all such α, β. Let us show that (8.30) holds (possibly after passing to a refinement of $\{U_\alpha\}$). We follow the standard argument. We have an exact sequence of sheaves

$$0\longrightarrow\mathbb{Z}\longrightarrow\mathcal{O}\xrightarrow{e^{2\pi i\cdot}}\mathcal{O}_*\longrightarrow 0,$$

that induces an exact sequences of cohomology groups

$$\ldots \longrightarrow H^1(c_{\mathfrak{a}}X,\mathbb{Z}) \longrightarrow H^1(c_{\mathfrak{a}}X,\mathcal{O}) \longrightarrow H^1\big(c_{\mathfrak{a}}X,\mathcal{O}_*\big) \xrightarrow{\sigma} H^2(c_{\mathfrak{a}}X,\mathbb{Z}) \longrightarrow \ldots \tag{8.31}$$

Clearly, if $c \in H^1\big(c_{\mathfrak{a}}X,\mathcal{O}_*\big)$ is trivial, then $\sigma(c) = 0$. We can repeat these arguments over a subset $Y := \bar{p}^{-1}(Y_0)$, where $Y_0 \subset X_0$ is open and is homotopically equivalent to X_0. In particular, over such Y we get an exact sequence of the form (8.31). Let $\sigma_Y : H^1\big(Y,\mathcal{O}_*\big) \longrightarrow H^2(Y,\mathbb{Z})$ denote the corresponding group homomorphism. Since X_0 is a Stein manifold, by Theorem 4.10(B) we have $H^1(c_{\mathfrak{a}}X,\mathcal{O}) = 0$. Hence, if $c \in H^1\big(c_{\mathfrak{a}}X,\mathcal{O}_*\big)$ is such that $\sigma(c) = 0$, then c is trivial. We denote by c_D the image in $H^1\big(c_{\mathfrak{a}}X,\mathcal{O}_*\big)$ of the 1-cocycle $C_D := \{d_{\alpha\beta} \in \mathcal{O}_*(U_\alpha \cap U_\beta)\}$, and by $c_{D,Y} \in H^1\big(Y,\mathcal{O}_*\big)$ the image of the restriction $C_{D,Y} := \{d_{\alpha\beta}|_Y \in \mathcal{O}_*(U_\alpha \cap U_\beta \cap Y\}$. If $D|_Y$ is equivalent to a principal divisor, then by the above argument $\sigma_Y(c_{D,Y}) = 0$. By the homotopy axiom for locally constant sheaves (see, e.g., [Bre, Ch. II.11]) we have $\sigma_Y(c_{D,Y}) = \sigma(c_D)$, hence c_D is trivial. Therefore, there exists a refinement $\{V_\gamma\}$ of cover $\{U_\alpha\}$ such that for the restrictions of functions $d_{\alpha\beta}$ to the corresponding V_γ there exists a presentation (8.30).

5.8.3. **Proof of Theorem 2.14. Assertion (1).** For an open $U \subset c_{\mathfrak{a}}X$, we denote by $\mathcal{O}_\sigma(U)$ the multiplicative semigroup of functions $h : U \to \mathbb{C}$ such that for every subset $V := \hat{\Pi}(V_0, K) \Subset U$, where $V_0 \subset X_0$ is open and simply connected, $K \subset \hat{G}_{\mathfrak{a}}$ is open, we have

(*i*) $h(\cdot,\omega)$ is in $\mathcal{O}(V_0)$ for every $\omega \in K$, and

(*ii*) $|h| \in C(V)$.

The next lemma is immediate.

Lemma 5.8.3. *If $h \in \mathcal{O}_\sigma(U)$, then $\iota^* h \in \mathcal{O}(\iota^{-1}(U))$, $|\iota^* h| \in C_{\mathfrak{a}}(\iota^{-1}(U))$.*

Let $(\mathcal{O}_\sigma)_*(V)$ denote the holomorphic functions in $\mathcal{O}_\sigma(V)$ that are nowhere zero; this is a multiplicative Abelian group.

Definition 5.8.4. A σ-divisor H is determined by an open cover $\{U_\alpha\}$ of $c_{\mathfrak{a}}X$ and functions $f_\alpha \in \mathcal{O}_\sigma(U_\alpha)$ such that $f_\alpha \not\equiv 0$ on any open subset of U_α, and $f_\alpha = d_{\alpha\beta} f_\beta$ on $U_\alpha \cap U_\beta \neq \varnothing$ for some $d_{\alpha\beta} \in (\mathcal{O}_\sigma)_*(U_\alpha \cap U_\beta)$, for all α, β.

The multiplicative semigroup of σ-divisor divisors is denoted by $\mathrm{Div}_\sigma(c_{\mathfrak{a}}X)$.

The divisors $D = \{(U_\alpha, f_\alpha)\}$, $D' = \{(U'_\beta, f'_\beta)\} \in \mathrm{Div}_\sigma(c_{\mathfrak{a}}X)$ are said to be equivalent if there exists a refinement $\{V_\gamma\}$ of both covers $\{U_\alpha\}$ and $\{U'_\beta\}$ and functions $c_\gamma \in (\mathcal{O}_\sigma)_*(V_\gamma)$ such that $f_\alpha|_{V_\gamma} = c_\gamma f'_\beta|_{V_\gamma}$ for $V_\gamma \subset U_\alpha \cap U'_\beta \neq \varnothing$, for all α, β.

The divisor $D_h = \{(c_{\mathfrak{a}}X, h)\} \in \mathrm{Div}_\sigma(c_{\mathfrak{a}}X)$, where $h \in \mathcal{O}_\sigma(c_{\mathfrak{a}}X)$, is called principal.

We have a monomorphism of semigroups

$$\mathrm{Div}(c_{\mathfrak{a}}X) \hookrightarrow \mathrm{Div}_\sigma(c_{\mathfrak{a}}X). \tag{8.32}$$

Assertion (1) would follow once we prove

Theorem 5.8.5. *Under the assumption of Theorem 2.14(1), every divisor $D \in \mathrm{Div}_\sigma(c_{\mathfrak{a}}X)$ is equivalent to a divisor $D' = \{(U_\alpha, f'_\alpha)\} \in \mathrm{Div}_\sigma(c_{\mathfrak{a}}X)$ such that $f'_\alpha/f'_\beta|_{\iota(X)\cap U_\alpha \cap U_\beta} \equiv 1$ for all α, β.*

Indeed, given a divisor $E \in \mathrm{Div}_{\mathfrak{a}}(X)$, by the results of Section 5.8.1 there exists a divisor $D \in \mathrm{Div}(c_{\mathfrak{a}}X)$ such that $E = \iota^* D$. In view of monomorphism (8.32), by Theorem 5.8.5 D is equivalent to a divisor D' in $\mathrm{Div}_\sigma(c_{\mathfrak{a}}X)$ whose pullback $E' = \iota^* D'$ is determined by a function in $\mathcal{O}_\sigma(X)$ (cf. Lemma 5.8.3), and is equivalent in $\mathrm{Div}(X)$ to E, so assertion (1) follows.

5.8.4. **Proof of Theorem 5.8.5.** We will need the following definitions and results.

DEFINITION 5.8.6. A divisor $D = \{(U_\alpha, f_\alpha)\} \in \mathrm{Div}_\sigma(c_{\mathfrak{a}}X)$ is called *cylindrical* if for every α we have $U_\alpha = \bar{p}^{-1}(U_{0,\alpha})$ for some open $U_{0,\alpha} \subset X_0$.

Proposition 5.8.7. *Every divisor $D = \{(U_\alpha, f_\alpha)\} \in \mathrm{Div}_\sigma(c_{\mathfrak{a}}X)$ is equivalent to a cylindrical divisor in $\mathrm{Div}_\sigma(c_{\mathfrak{a}}X)$.*

Let $\mathbb{Z}_d$ be the additive sheaf associated to the presheaf of functions defined similarly to $\mathcal{O}_\sigma(U)$, $U \subset c_{\mathfrak{a}}X$ is open, but with condition (ii) omitted, and (i) replaced with the assumption that $f(\cdot,\omega)$ is identically equal to an integer, for every $\omega \in K$. Let $\mathbb{R}_d := \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}_d$.

Proposition 5.8.8. *Let $U_0 \subset X_0$ be open, $U := \bar{p}^{-1}(U_0)$. Let $D = \{(U_\alpha, f_\alpha)\} \in \mathrm{Div}_\sigma(U)$ be a cylindrical divisor such that*

(i) the sets $U_{0,\alpha} \subset U_0$ are open and simply connected, and

(ii) $\mathcal{U} := \{U_{0,\alpha}\}$ is an open cover of of U_0 such that the connected components of the intersection of any two sets in $\mathcal{U}$ are simply connected, and the intersection of any three sets in $\mathcal{U}$ is empty.

Then divisor D is equivalent to a cylindrical divisor $D' = \{(U_\alpha, f'_\alpha)\} \in \mathrm{Div}_\sigma(c_{\mathfrak{a}}X)$ such that $f'_\alpha/f'_\beta|_{\iota(X)\cap U_\alpha \cap U_\beta} \equiv 1$ for all α, β.

We will also need the following version of Proposition 5.8.8.

Proposition 5.8.9. *Let $U_0 \subset X_0$ be open and simply connected, $U := \bar{p}^{-1}(U_0)$. Let $D = \{(U_\alpha, f_\alpha)\} \in \mathrm{Div}_\sigma(U)$ be a cylindrical divisor such that*

(i) the sets $U_{0,\alpha} \subset U_0$ are open and simply connected, and

(ii) $\mathcal{U} := \{U_{0,\alpha}\}$ is an open cover of of U_0 such that the connected components of the intersection of any two sets in $\mathcal{U}$ are simply connected, and the intersection of any three sets in $\mathcal{U}$ is empty.

Then there exists a function $h \in \mathcal{O}_\sigma(U)$ such that D is equivalent to $D_h \in \mathrm{Div}_\sigma(U)$.

Corollary 5.8.10. *Let $U_0 \subset X_0$ be open and simply connected, $U := \bar{p}^{-1}(U_0)$, let $D = \{(U_\alpha, f_\alpha)\} \in \mathrm{Div}_\sigma(U)$ be a cylindrical divisor. Then D is equivalent to a cylindrical divisor $D' = \{(U_\alpha, f'_\alpha)\} \in \mathrm{Div}_\sigma(c_{\mathfrak{a}}X)$ such that $f'_\alpha/f'_\beta|_{\iota(X)\cap U_\alpha \cap U_\beta} \equiv 1$ for all α, β.*

We prove Propositions 5.8.7, 5.8.8 and Corollary 5.8.10 in the next section.

We now complete the proof of Theorem 5.8.5. Since X_0 is a Riemann surface, it admits a strong deformation retract $S_t : X_0 \to X_0$ ($t \in [0,1]$) to a 1-dimensional CW-complex $\Gamma \subset X_0$. By definition, $S_0 = \mathrm{Id}$, $S_1(X_0) = \Gamma$. Since Γ is locally contractible, there exists an open cover $\mathcal{V} = \{V_\alpha\}$ of Γ by contractible open sets, such that the intersection of any two sets in $\mathcal{V}$ is contractible, and the intersection of any three sets in $\mathcal{V}$ is empty. We define $U_{0,\alpha} := S_0^{-1}(V_{0,\alpha})$. Then $\mathcal{U} := \{U_{0,\alpha}\}$ forms an open cover of X_0 that satisfies conditions of Proposition 5.8.8. The proof now follows by consequtive application of Propostion 5.8.7, Corollary 5.8.10 (to each $U_{0,\alpha}$) and Proposition 5.8.8.

5.8.5. **Proofs of Propositions 5.8.7–5.8.9 and Corollary 5.8.10.** Let $(\mathcal{O}_\sigma)_*$ denote the multiplicative sheaf associated to the presheaf of functions $(\mathcal{O}_\sigma)_*(U)$, $U \subset c_{\mathfrak{a}}X$ open. We denote by $\mathcal{O}_{\mathrm{Re}}$ the additive sheaf on $c_{\mathfrak{a}}X$ associated to the presheaf of functions that are defined similarly to the functions in $\mathcal{O}_\sigma(U)$, $U \subset c_{\mathfrak{a}}X$ open, but now instead of condition (ii) we require that $\mathrm{Re}\, f \in C(V)$.

(a) We have an exact sequence

$$0 \longrightarrow \mathbb{Z}_d \xrightarrow{\ i\ } \mathcal{O}_{\mathrm{Re}} \xrightarrow{\ e^{2\pi\cdot}\ } (\mathcal{O}_\sigma)_* \longrightarrow 0, \tag{8.33}$$

where i is the composition of the multiplication by $\sqrt{-1}$ and inclusion.

(b) Using Lemma 5.8.2, below we may assume that $\{U_\alpha\}$ is at most countable, and each $U_\alpha = \hat{\Pi}(U_{0,\alpha}, K_\alpha)$ (cf. (4.14)), where $U_{0,\alpha} \subset X_0$ is open and simply connected, $K_\alpha \in \mathfrak{Q}$ (cf. (4.12)). Clearly, $\{U_{0,\alpha}\}$ is an open cover of X_0, and $\{K_\alpha\}$ is an open cover of $\hat{G}_{\mathfrak{a}}$.

(c) We will need

Lemma 5.8.11. *Let $U := \hat{\Pi}(U_0, K)$, where $U_0 \subset X_0$ is open and simply connected, and $K \in \mathfrak{Q}$ (cf. (4.12)). Then $H^k(U, \mathbb{R}_d) = 0$, $k \geq 1$.*

Proof of Lemma 5.8.11. We may identify U with $U_0 \times K$ (cf. Section 4.5). It is not difficult to see that space $U_0 \times K$ is paracompact. Therefore, it suffices to show that, given an at most countable open cover $\{W_l\}$ of $U_0 \times K$, every k-cocycle $\sigma = \{\sigma_I \in \Gamma(W_{l_1} \cap \cdots \cap W_{l_k}, \mathbb{R}_d) : I = (l_1, \dots, l_k)\} \in Z^k(\{W_l\}, \mathbb{R}_d)$ (cf. Section 5.1.1) can be presented in the form $\sigma = \delta\eta$ for some $k-1$-cochain $\eta \in C^{k-1}(\{W_l\}, \mathbb{R}_d)$. Indeed, let $i_\omega : U_0 \to U_0 \times K$, $i_\omega(x) := (x, \omega)$ ($x \in U_0, \omega \in K$) be the natural inclusion. For each $\omega \in K$ we denote $W_{l,\omega} := i_\omega^{-1}(W_l)$ for all l (some of these sets will be empty). This is an open cover of U_0. Next, define $\sigma_{I,\omega} := i_\omega^* \sigma_I$ for all I (by definition, if $W_{l_1,\omega} \cap \cdots \cap W_{l_k,\omega} = \varnothing$, then $\sigma_{I,\omega} := 0$). Then $\sigma_\omega := \{\sigma_{I,\omega}\} \in Z^k(\{W_{l,\omega}\}, \mathbb{R})$. Now, since U_0 is contractible, we have $H^k(U_0, \mathbb{R}) = 0$, $k \geq 1$. We may assume without loss of generality that the connected components of any intersection $W_{l,\omega} \cap \cdots \cap W_{l_{k-1},\omega}$ are simply connected. Hence, using Leray theorem (see, e.g., [Gun3]) we obtain that for every $\omega \in K$ there exists a $k-1$-cochain $\eta_\omega = \{\eta_{J,\omega} \in \Gamma(W_{l,\omega} \cap \cdots \cap W_{l_{k-1},\omega}, \mathbb{R}) : J = (l_1, \dots, l_{k-1}\} \in C^{k-1}(\{W_{l,\omega}\}, \mathbb{R})$ such that $\sigma_\omega = \delta\eta_\omega$, where operator $\delta = \delta_{\{W_{l,\omega}\}}$ has the same formal definition as $\delta_{\{W_{l,\omega}\}}$ since there is a bijective correspondence between the elements of covers $\{W_l\}$ and $\{W_{l,\omega}\}$. We define $\eta_J(z, \omega) := \eta_{J,\omega}(z)$ for all $J = (l_1, \dots, l_{k-1})$, $\omega \in K$ and $z \in W_{l_1,\omega} \cap \cdots \cap W_{l_{k-1},\omega}$. It is immediate from the definition of sheaf $\mathbb{R}_d$ that $\eta := \{\eta_J\}$ is a $k-1$-cochain in $C^{k-1}(\{W_l\}, \mathbb{R}_d)$. Further, since equality $\sigma_\omega = \delta\eta_\omega$ holds for all $\omega \in K$, we obtain that $\sigma = \delta\eta$, as required. □

Proof of Proposition 5.8.7. We have to show that D is equivalent to a divisor $H = \{(V_\beta, h_\beta)\} \in \mathrm{Div}_\sigma(c_{\mathfrak{a}}X)$, where $V_\beta = \bar{p}^{-1}(V_{0,\beta})$ for some open simply connected sets $V_{0,\beta} \subset X_0$.

Let $x_0 \in X_0$. Since $\bar{p}^{-1}(x_0) \cong \hat{G}_{\mathfrak{a}}$ is compact, there exist finitely many sets $U_{\alpha_i} = \hat{\Pi}(U_{0,\alpha_i}, K_{\alpha_i})$ (cf. (b)), $1 \leq i \leq m$, such that $\bar{p}^{-1}(x_0) \subset \cup_{i=1}^m U_{\alpha_i}$. Hence, there exists an open simply connected neighbourhood $V_0 \subset \cap_{i=1}^m U_{0,\alpha_i}$ of x_0 such that $V := \bar{p}^{-1}(V_0) \subset \cup_{i=1}^m U_{\alpha_i}$.

We set $V_i := U_{\alpha_i} \cap V$, $K_i := K_{\alpha_i}$, and $f_i := f_{\alpha_i}|_{V_i} \in \mathcal{O}_\sigma(V_i)$ $(1 \leq i \leq m)$.

Since $x_0 \in X_0$ was chosen arbitrarily, it suffices to prove that for the restriction $D|_V = \{(V_i, f_i)\}$ there exists a function $h \in \mathcal{O}_\sigma(V)$ such that $D|_V$ is equivalent in $\mathrm{Div}_\sigma(V)$ to the principal divisor $D_h \in \mathrm{Div}_\sigma(V)$. (The definition of $\mathrm{Div}_\sigma(V)$ is completely analogous to the definition of $\mathrm{Div}_\sigma(c_{\mathfrak{a}}X)$.) We will need

Lemma 5.8.12. *Let $W := \hat{\Pi}(W_0, K)$, where $W_0 \subset X_0$ is open and simply connected, and $K \in \mathfrak{Q}$ (cf. (4.12)). Then $H^k(W, \mathbb{Z}_d) = 0$, $k \geq 1$.*

The proof is similar to the proof of Lemma 5.8.11.

It follows from Lemma 5.8.12 (with $W := V_i$, $1 \leq i \leq m$) that the exact sequence of cohomology groups induced by the short exact sequence (8.33) has form

$$0 \longrightarrow \Gamma(V_i, \mathbb{Z}_d) \longrightarrow \Gamma(V_i, \mathcal{O}_{\mathrm{Re}}) \longrightarrow \Gamma\big(V_i, (\mathcal{O}_\sigma)_*\big) \longrightarrow 0 \longrightarrow \dots \tag{8.34}$$

($1 \le i \le m$). An argument similar to the one in the proof of Lemma 5.8.11 shows that we may replace set V_i in the formulation of the lemma and in (8.34) with an intersection of sets V_i, $1 \le i \le m$. Hence, we have an exact sequence of cochain complexes

$$0 \longrightarrow C^k(\{V_i\}, \mathbb{Z}_d) \longrightarrow C^k(\{V_i\}, \mathcal{O}_{\mathrm{Re}}) \longrightarrow C^k\big(\{V_i\}, (\mathcal{O}_\sigma)_*\big) \longrightarrow 0, \quad k \ge 0. \tag{8.35}$$

The exact sequence (8.35) induces an exact sequence of Čech cohomology groups corresponding to cover $\{V_i\}$

$$\dots \longrightarrow H^1(\{V_i\}, \mathbb{Z}_d) \longrightarrow H^1(\{V_i\}, \mathcal{O}_{\mathrm{Re}}) \xrightarrow{e^{2\pi\cdot}} H^1\big(\{V_i\}, (\mathcal{O}_\sigma)_*\big) \longrightarrow H^2(\{V_i\}, \mathbb{Z}_d) \longrightarrow \dots \tag{8.36}$$

An argument similar to the one in the proof of Lemma 5.8.11 yields $H^2(\{V_i\}, \mathbb{Z}_d) = 0$. By definition, functions f_i satisfy $f_i = d_{ij} f_j$ on $V_i \cap V_j \ne \varnothing$ for some $d_{ij} \in (\mathcal{O}_\sigma)_*(V_i \cap V_j)$, and $d_D := \{d_{ij}\}$ is a 1-cocycle. We have to show that there exist $d_i \in (\mathcal{O}_\sigma)_*(V_i)$ such that $d_{ij} = d_j d_i^{-1}$; then we will replace f_i with $f_i d_i$, thus obtaining a globally defined function $h := f_i d_i$ in V, as needed. Since (8.36) is exact and $H^2(\{V_i\}, \mathbb{Z}_d) = 0$, there exists a 1-cocycle $\{g_{ij} \in \mathcal{O}_{\mathrm{Re}}(V_i \cap V_j)\}$ such that $d_{ij} = e^{2\pi g_{ij}}$, for all $1 \le i, j \le m$. Let us show that there exists a 0-cochain $\{g_i \in \mathcal{O}_{\mathrm{Re}}(V_i)\}$ such that $g_{ij} = g_j - g_i$ on $V_i \cap V_j \ne \varnothing$, for all $1 \le i, j \le m$. Once we find such g_i, we will set $d_i = e^{2\pi g_i} \in (\mathcal{O}_\sigma)_*(V_i)$, $1 \le i \le m$, thus completing the proof.

We construct functions g_i, $1 \le i \le m$, as follows. Let $\{\rho_i\}_{i=1}^m \subset C(\hat{G}_{\mathfrak{a}})$ be a partition of unity subordinate to open cover $\{K_i\}_{i=1}^m$. Now, we define $g_i := \sum_{i,j=1}^m \rho_j g_{ij}$ on V_i ($1 \le i \le m$). Clearly, the 0-cochain $\{g_i\}$ gives a resolution of the 1-cocycle $\{g_{ij}\}$. Under the identification of $V = \bar{p}^{-1}(V_0)$ and $V_0 \times \hat{G}_{\mathfrak{a}}$ (cf. Section 4.5), for every $1 \le i \le m$ and every $\omega \in \hat{G}_{\mathfrak{a}}$ the function $\rho_i(\cdot, \omega)$ is constant and hence is holomorphic on V_0; also, functions ρ_i viewed as functions on V, are continuous, hence $\rho_i \in \mathcal{O}(V)$ ($1 \le i \le m$). It follows that for every $\omega \in K_i$ we have $g_i(\cdot, \omega) \in \mathcal{O}(V_0)$, and the real part of g_i is continuous on V, as functions ρ_i are real-valued and the real parts of functions g_{ij} are continuous on V. Therefore, $g_i \in \mathcal{O}_{\mathrm{Re}}(V_i)$, for all $1 \le i \le m$, as needed. □

Proof of Proposition 5.8.8. We will use notation introduced in the proof of Proposition 5.8.7. By Lemma 5.8.12 (there we take $W := U_\alpha$) the short exact sequence (8.33) induces an exact sequence of the form

$$0 \longrightarrow \Gamma(U_\alpha, \mathbb{Z}_d) \longrightarrow \Gamma(U_\alpha, \mathcal{O}_{\mathrm{Re}}) \longrightarrow \Gamma\big(U_\alpha, (\mathcal{O}_\sigma)_*\big) \longrightarrow 0 \longrightarrow \dots, \quad \text{for all } \alpha.$$

Thus, we obtain an exact sequence of chain complexes

$$0 \longrightarrow C^l(\{U_\alpha\}, \mathbb{Z}_d) \longrightarrow C^l(\{U_\alpha\}, \mathcal{O}_{\mathrm{Re}}) \longrightarrow C^l\big(\{U_\alpha\}, (\mathcal{O}_\sigma)_*\big) \longrightarrow 0, \quad l \ge 0. \tag{8.37}$$

In turn, sequence (8.37) induces an exact sequence of Čech cohomology groups

$$\dots \longrightarrow H^1(\{U_\alpha\}, \mathbb{Z}_d) \longrightarrow H^1(\{U_\alpha\}, \mathcal{O}_{\mathrm{Re}}) \xrightarrow{e^{2\pi\cdot}} H^1\big(\{U_\alpha\}, (\mathcal{O}_\sigma)_*\big) \longrightarrow H^2(\{U_\alpha\}, \mathbb{Z}_d) \longrightarrow \dots \tag{8.38}$$

We may assume without loss of generality that the connected components of any intersection of sets $U_{0,\alpha}$ are simply connected. Hence, by Leray theorem (see, e.g., [Gun3]) and Lemma 5.8.12 $H^2(\{U_\alpha\}, \mathbb{Z}_d) = 0$. Now, it follows from (8.38) that

$$H^1(\{U_\alpha\}, \mathcal{O}_{\mathrm{Re}}) \xrightarrow{e^{2\pi\cdot}} H^1\big(\{U_\alpha\}, (\mathcal{O}_\sigma)_*\big) \quad \text{is surjective.} \tag{8.39}$$

Let $d = \{d_{\alpha\beta}\} \in Z^1(\{U_\alpha\}, (\mathcal{O}_\sigma)_*)$ be the 1-cocycle determined by divisor D. Using (8.39), we obtain that (possibly after replacing divisor D with an equivalent one) there exists a 1-cocycle $c = \{c_{\alpha\beta}\} \in Z^1(\{U_\alpha\}, \mathcal{O}_{\mathrm{Re}})$ such that $d_{\alpha\beta} = e^{2\pi c_{\alpha\beta}}$, for all α, β. Let us show that there exists a 1-cocycle $c' = \{c'_{\alpha\beta}\} \in Z^1(\{U_\alpha\}, \mathcal{O}_{\mathrm{Re}})$ that is equivalent to c, i.e., $c_{\alpha\beta} - c'_{\alpha\beta} = c_\beta - c_\alpha$ for some

0-cochain $\{c_\alpha\} \in C^0(\{U_\alpha\}, \mathcal{O}_{\rm Re})$, and is such that $c'_{\alpha\beta} = 0$ on $\iota(X) \cap U_\alpha \cap U_\beta$ for all α, β. Once we find such a cocycle, we can replace functions f_α determining divisor D by $f'_\alpha := f_\alpha e^{2\pi c_\alpha}$, so that $f'_\alpha = d'_{\alpha\beta} f'_\beta$ for all α, β, where $d'_{\alpha\beta} := e^{2\pi c'_{\alpha\beta}}$. Since $d'_{\alpha\beta} = f'_\alpha / f'_\beta \equiv 1$ on $\iota(Y) \cap U_\alpha \cap U_\beta$, the divisor $D' := \{(U_\alpha, f'_\alpha)\}$ is the one required in Assertion (1).

Now, to find $\{c'_{\alpha\beta}\}$, we will use the following result.

Lemma 5.8.13. *There exist* $\tilde c \in Z^1(\{U_\alpha\}, \mathcal{O})$, $r \in Z^1(\{U_\alpha\}, \mathbb{R}_d)$ *such that* $c = \tilde c + ir$.

Proof of Lemma 5.8.13. Let $C^j_{\alpha\beta}$ denote the connected components of $U_{0,\alpha} \cap U_{0,\beta}$. By our construction, these are simply connected open subsets of U_0. We identify U_α and U_β with $U_{0,\alpha} \times \hat G_{\mathfrak a}$ and $U_{0,\alpha} \times \hat G_{\mathfrak a}$, respectively, and $U_\alpha \cap U_\beta$ with $(U_{0,\alpha} \cap U_{0,\beta}) \times \hat G_{\mathfrak a}$ (cf. Section 4.5).

Let us fix some points $x^j_{\alpha\beta} \in C^j_{\alpha\beta}$, for all $1 \le j \le t$. We define

$$r_{\alpha\beta} := \mathrm{Im}\; c_{\alpha\beta}(x_j, \cdot) \quad \text{ on } C^j_{\alpha\beta} \times \hat G_{\mathfrak a}, \text{ for all } j.$$

It follows from (ii) that $r = \{r_{\alpha\beta}\}$ is a 1-cocylce in $Z^1(\{U_\alpha\}, \mathbb{R}_d)$. Further, we set $\tilde c_{\alpha\beta} := c_{\alpha\beta} - ir_{k-1,k}$. Let us show that $\tilde c_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)$, that is, the restriction of $\tilde c_{k-1,k}$ to each $C^j_{\alpha\beta}$ belongs to $\mathcal{O}(C^j_{\alpha\beta} \times \hat G_{\mathfrak a})$. Indeed, for each fixed $\omega \in \hat G_{\mathfrak a}$ the restriction of $\tilde c_{\alpha\beta}(\cdot, \omega)$ to $C^j_{\alpha\beta}$ is in $\mathcal{O}(C^j_{\alpha\beta})$, hence we only need to prove, for every j, the continuity of map

$$\hat G_{\mathfrak a} \ni \omega \mapsto \tilde c_{\alpha\beta}(\cdot, \omega)|_{C^j_{\alpha\beta}} \in C(C^j_{\alpha\beta}), \tag{8.40}$$

where space $C(C^j_{\alpha\beta})$ of continuous functions on $C^j_{\alpha\beta}$ is endowed with the topology of uniform convergence on compact subsets. By definition, map

$$\omega \mapsto \mathrm{Re}\; \tilde c_{\alpha\beta}(\cdot, \omega) \in C(C^j_{\alpha\beta}) \tag{8.41}$$

is continuous. Since $\mathrm{Im}\; \tilde c_{\alpha\beta}(\cdot, \omega) \in C(C^j_{\alpha\beta})$ is a conjugate harmonic function of $\mathrm{Re}\; \tilde c_{\alpha\beta}(\cdot, \omega)$, for all ω, the continuity of (8.41) follows from the integral formula presentation for conjugate harmonic functions, and the condition that $\mathrm{Im}\; \tilde c_{\alpha\beta}(x_j, \omega) = 0$ for all $\omega \in \hat G_{\mathfrak a}$. This implies the continuity of (8.40), as needed. □

We now complete the proof of Proposition 5.8.8. We have $H^1(\{U_\alpha\}, \mathcal{O}) = 0$ (see, e.g. [ZK]). Therefore, there exists a 0-cochain $\{\tilde c_\alpha\} \in C^0(\{U_\alpha\}, \mathcal{O})$ such that $\tilde c_{\alpha\beta} = \tilde c_\beta - \tilde c_\alpha$. Since $H^1(X, \mathbb{R}) = 0$, we obtain (using Leray theorem) that there exist a 0-cochain $\{r_\alpha\} \in C^0(\{U_\alpha\}, \mathbb{R}_d)$ such that $r_\alpha = 0$ on $U_\alpha \setminus \iota(X)$, and $r_\beta - r_\alpha = r_{\alpha\beta}$ on $\iota(X) \cap U_\alpha \cap U_\beta$. We define $c_\alpha := \tilde c_\alpha + ir_\alpha$, for all α, and set $c'_{\alpha\beta} := c_{\alpha\beta} - c_\beta + c_\alpha$. This completes the proof. □

Proof of Proposition 5.8.9. The proof follows closely the proof of Proposition 5.8.8, except for the last step, where, instead of replacing the 1-cocycle $c_{\alpha\beta}$ with an equivalent 1-cocycle we resolve it (applying Lemma 5.8.11 to $\{r_{\alpha\beta}\}$). □

Proof of Corollary 5.8.10. (1) First, let us assume the following:

(i) Cover $\{U_\alpha\}$ of U_0 is finite, i.e. $\{U_\alpha\} = \{U_{\alpha_i}\}_{i=1}^m$. We denote $U_{0,i} := U_{0,\alpha_i}$, $U_i := U_{\alpha_i}$ $(= \bar p^{-1}(U_{0,\alpha_i}))$, $f_i := f_{\alpha_i}$, $1 \le i \le m$.

(ii) We have ordered sets U_i in such a way that the sets $V_0^k := \cup_{i=1}^k U_{0,i}$, $1 \le k \le m$, are simply connected, and the connected components of the intersection of $U_{0,k}$ and V^{k-1} are simply connected, for all $2 \le k \le m$.

We denote $V^k := \bar p^{-1}(V_{0,k})$. Clearly, $V^m = U_0$. Next, we proceed by induction over k, and find functions $h_k \in \mathcal{O}_\sigma(V^k)$, $1 \le k \le m$, such that $D|_{V^k} = \{(U_i, f_\beta) : 1 \le i \le k\}$ is equivalent to D_{h_k}. Once such functions are found, we can set $h := h_m$ to complete the proof.

For $k = 1$ we take $h_1 := f_1$. Suppose that we have constructed h_{k-1}, let us find h_k. Since $D|_{V^{k-1}}$ is equivalent to $D_{h_{k-1}}$, we obtain that $\tilde{D}_k := \{(V^{k-1}, h_{k-1}), (U_k, f_k)\}$ is a σ-divisor on V^k equivalent to $D|_{V^k}$. Since the open cover $\{V^{k-1}, U_k\}$ of V^k satisfies the conditions of Proposition 5.8.9, there exists a function $h_k \in \mathcal{O}_\sigma(V^k)$ such that $\tilde{D}_k$ is equivalent to D_{h_k}. This completes the proof of Case (1).

(2) Consider a general cover $\{U_\alpha\}$. Let us fix some $x_0 \in U_0$. Let $V_0 \subset U_0$ be a neighbourhood of x_0 such that $\bar{V}_0$ is contained in some coordinate chart $\tilde{V}_0$ of U_0, and let $\varphi : \tilde{V}_0 \to \mathbb{C}$ be the corresponding biholomorphic coordinate map such that $\varphi(x_0) = 0$. We may assume without loss of generality that $\varphi(\bar{V}_0) = \bar{B}_r = \{z \in \mathbb{C} : |z| \le r\}$. Furthermore, since U_0 is simply connected, we can choose V_0 in such a way that there exists a strong deformation retract $S_t : U_0 \to U_0$ of U_0 to $\bar{V}_0$. By definition, $S_0 = \mathrm{Id}$, $S_1(U_0) = \bar{V}_0$. Now, let $\{\Lambda_j\}_{j=0}^m$ be an open cover of $\bar{B}_r \setminus \{0\}$ by open cones Λ_j with vertices at 0. We set $\Omega_{0,j} := (\varphi \circ S_0)^{-1}(\Lambda_j), \quad 0 \le j \le m$. Then $\{\Omega_{0,j}\}_{j=0}^m$ is an open cover of $U_0 \setminus \{x_0\}$.

We define $W_0' := \cup_{j=1}^m \Omega_{0,j}$ (i.e. we exclude $\Omega_{0,0}$). This is an open simply connected subset of U_0. Next, we can choose a simply connected open neighbourhood W_0'' of $\bar{\Omega}_{0,0}$ so that $W_0' \cap W_0''$ is simply connected. Then $\{W_0', W_0''\}$ forms an open cover of U_0 that satisfies conditions of Proposition 5.8.9 (note that since U_0 is simply connected, we have $H^1(\bar{p}^{-1}(U_0), \mathbb{R}_d) = 0$ by Lemma 5.8.11). Thus, if we can show that D is equivalent to a cylindrical σ-divisor $\{(W', h'), (W'', h'')\}$, where $W' := \bar{p}^{-1}(W_0')$, $W'' := \bar{p}^{-1}(W_0'')$, and $h' \in \mathcal{O}_\sigma(W')$, $h'' \in \mathcal{O}_\sigma(W'')$, then applying Proposition 5.8.9 we can obtain the required function h. Indeed, note that the intersection $\Omega_{0,j}$ $(0 \le j \le m)$ with any relatively compact subset of U_0 is relatively compact in U_0. Hence, for every fixed $0 \le j_0 \le m$ there exists an open cover $\{\Omega^i_{0,j_0}\}$ that satisfies conditions of Proposition 5.8.9, and is such that each set Ω_{0,j_0} is relatively compact in U_0. Hence, using the result of Case (1) (clearly, every relatively compact subset of U_0 can be covered by open sets satisfying (i),(ii)), we may assume without loss of generality that sets $U_{0,\alpha}$ in the definition of divisor D are sufficiently large, so that $\{\Omega^i_{0,j}\}$ is a refinement of the open cover $\{U_{0,\alpha}\}$.

Now, for each fixed $1 \le j_0 \le m$ we apply Proposition 5.8.9 to the open cover $\{\Omega^i_{0,j_0}\}$ of Ω_{0,j_0}, thus obtaining a function $f_{j_0} \in \mathcal{O}_\sigma(\Omega_{j_0})$, where $\Omega_{j_0} := \bar{p}^{-1}(\Omega_{0,j_0})$, such that $D|_{\Omega_{j_0}}$ is equivalent to $D_{f_{j_0}} \in \mathrm{Div}_\sigma(\Omega_{j_0})$.

The same argument applied to the open cover of W_0'', obtained from the open cover$\{\Omega^i_{0,0}\}$ of Ω_0 by enlarging sets $\Omega^i_{0,0}$ slightly, implies that there exists a function $h'' \in \mathcal{O}_\sigma(W'')$ such that $D|_{W''}$ is equivalent to $D_{h_0} \in \mathrm{Div}_\sigma(W'')$.

To obtain function h', note that the open cover $\{\Omega_{0,j}\}_{j=1}^m$ of W_0' itself satisfies conditions of Proposition 5.8.9. Therefore, there exists a function $h_1 \in \mathcal{O}_\sigma(W')$ such that divisor $\{(\Omega_j, f_j)\}_{j=1}^m$ is equivalent to $D_{h'} \in \mathrm{Div}_\sigma(W')$. The proof is complete. □

Assertion (2). We will use notation introduced in the proof of Assertion (1), as well as notation and results of Section 4.1 and Example 4.5. It suffices to show that, for a given open simply connected subset $U_0 \subset X_0$, the divisor in $\mathrm{Div}\big(p^{-1}(U_0)\big)$ determined by the restriction $f|_{p^{-1}(U_0)}$ is equivalent to a divisor in $\mathrm{Div}_{\mathfrak{a}}\big(p^{-1}(U_0)\big)$. In what follows, we identify $p^{-1}(U_0)$ with $U := U_0 \times G$, and $\bar{p}^{-1}(U_0)$ with $\hat{U}_{\mathfrak{a}} := U_0 \times \hat{G}_{\mathfrak{a}}$ (cf. Section 4.5). Denote $\hat{U}_{\ell_\infty} := U_0 \times \hat{G}_{\ell_\infty}$.

We conduct our proof is two steps.

(1) First, let us show that there exists a function $F : \hat{U}_{\mathfrak{a}} \to \mathbb{C}$ such that $|F| \in C(\hat{U}_{\mathfrak{a}})$, $0 \not\equiv F(\cdot, \omega) \in \mathcal{O}(U_0)$ for all $\omega \in \hat{G}_{\mathfrak{a}}$, and $f|_U = j_{\mathfrak{a}}^* F$. Since $|f|_U| \in C_{\mathfrak{a}}(U)$, there is a function $M_f \in C(\hat{U}_{\mathfrak{a}})$ such that $|f|_U| = j_{\mathfrak{a}}^* M_f$. Further, we have $f|_U \in \mathcal{O}_{\ell_\infty}(U)$, hence there exists a function $S_f \in \mathcal{O}(\hat{U}_{\ell_\infty})$ such that $f|_U = j_{\ell_\infty}^* S_f$. Note that $S_f(\cdot, \eta) \not\equiv 0$ for all $\eta \in \hat{G}_{\ell_\infty}$, for

otherwise, since S_f is continuous, there would exist a net $\{g_\alpha\} \subset G$, $j_{\ell_\infty}(g) \to \eta_0 \in \hat{G}_{\ell_\infty}$, such that the sequence $\{S_f(\cdot, j_{\ell_\infty}(g_\alpha))\}$ would converge to zero uniformly on compact subsets of U_0, which is not possible by our assumption. We define function $F : \hat{U}_{\mathfrak{a}} \to \mathbb{C}$ by $F(x,\omega) := S_f\big(x, \lambda(\omega)\big)$, $(x,\omega) \in \hat{U}_{\mathfrak{a}}$. We have $|F| = M_f$. Indeed, let $\tilde{M}_f := (\mathrm{Id} \times \kappa)^* M_f \in C(U_0 \times \hat{G}_{\ell_\infty})$. By definition, $\big|S_f\big(x, j_{\ell_\infty}(g)\big)\big| = \tilde{M}_f\big(x, j_{\ell_\infty}(g)\big) = M_f\big(x, j_{\mathfrak{a}}(g)\big)$, $(x,g) \in U$. Since $j_{\ell_\infty}(G)$ is dense in $\hat{G}_{\ell_\infty}$ (cf. Section 4.1) and both functions $|S_f|$ and $\tilde{M}_f$ are continuous, we obtain that $|S_f| = \tilde{M}_f$. By definition, we have $\tilde{M}\big(x, \lambda(\omega)\big) = M_f(x,\omega)$, $(x,\omega) \in \hat{U}_{\mathfrak{a}}$, hence $|F| = M_f$. It follows that $|F| \in C(\hat{U}_{\mathfrak{a}})$, as required.

(2) Now, let $D_F \in \mathrm{Div}_\sigma(\hat{U}_{\mathfrak{a}})$ be the σ-divisor determined by function F, and $E_f \in \mathrm{Div}(U)$ be the divisor determined by $f|_U$. By our construction, $(\mathrm{Id} \times j_{\mathfrak{a}})^* D_H = E_f$. It remains to show that D_H is equivalent in $\mathrm{Div}_\sigma(\hat{U}_{\mathfrak{a}})$ to a divisor $D \in \mathrm{Div}(\hat{U}_{\mathfrak{a}})$; then E_f would be equivalent in $\mathrm{Div}(X)$ to $E := (\mathrm{Id} \times j_{\mathfrak{a}})^* D \in \mathrm{Div}_{\mathfrak{a}}(U)$, as needed. For a given $x \in U_0$ we denote $I_x := \{\omega \in \hat{G}_{\mathfrak{a}} : F(x,\omega) \neq 0\}$. Since $|F|$ is continuous, each set I_x is open. We have $\hat{G}_{\mathfrak{a}} = \cup_{x \in U_0} I_x$. Indeed, for otherwise there exists $\omega_0 \in \hat{G}_{\mathfrak{a}}$ such that $F(\cdot, \omega_0) \equiv 0$, which is a contradiction. Since $\hat{G}_{\mathfrak{a}}$ is compact, there exist points x_i, $1 \le i \le m$, such that $\hat{G}_{\mathfrak{a}} = \cup_{i=1}^m I_{x_i}$. We fix some $1 \le i_0 \le m$, and define

$$F_{i_0}(x,\omega) := F(x,\omega) e^{-\mathrm{Arg} F(x_{i_0},\omega)}, \quad x \in U_0, \quad \omega \in I_{x_{i_0}}. \tag{8.42}$$

Since $F(x_{i_0}, \omega) \neq 0$ for all $\omega \in I_{x_{i_0}}$, this function is well defined. By definition, $F_{i_0}(\cdot, \omega) \in \mathcal{O}(U_0)$ for every $\omega \in I_{x_{i_0}}$.

Lemma 5.8.14. *$F_{i_0} \in C(U_0 \times I_{x_{i_0}})$.*

Proof of Lemma. We fix some $\omega_0 \in I_{x_{i_0}}$. Let $\{\omega_\alpha\} \subset I_{x_{i_0}}$ be a net such that $\omega_\alpha \to \omega_0$. Using Montel theorem, it suffices to prove that all partial limits $\{c_\beta\} \subset \mathcal{O}(U_0)$ of $\{F_{i_0}(\cdot, \omega_\alpha)\} \subset \mathcal{O}(U_0)$ coincide with $F_{i_0}(\cdot, \omega_0)$. Indeed, since $|F_{i_0}| = |F|$ is continuous on $U_0 \times I_{x_{i_0}}$, we obtain that a partial limit c_β differs from $F_{i_0}(\cdot, \omega_0)$ by a constant multiple of modulus 1. By (8.42) $F_{i_0}(x_{i_0}, \omega) \in \mathbb{R}$ for all ω, hence this multiple must be equal to 1, i.e., $c_\beta = F_{i_0}(\cdot, \omega_0)$ for all β, as needed. □

It follows that $D := \{(U_0 \times I_{x_i}, F_i)\}$, $1 \le i \le m$, is a divisor in $\mathrm{Div}(\hat{U}_{\mathfrak{a}})$. By our construction, D is equivalent to D_H in $\mathrm{Div}_\sigma(\hat{U}_{\mathfrak{a}})$, which completes the proof.

5.9. Proof of Theorem 4.11

Lemma 5.9.1. *Let $U_0 \subset X_0$, $K \subset \hat{G}_{\mathfrak{a}}$ be open, $f \in \mathcal{O}(U_0 \times K)$ be such that $\nabla_z f(z,\eta) \neq 0$ for all $(z,\eta) \in Z_f := \{(z,\eta) \in U_0 \times K : f(z,\eta) = 0\}$.*

If $g \in \mathcal{O}(U_0 \times K)$ vanishes on Z_f, then $h := g/f \in \mathcal{O}(U_0 \times K)$.

(The proof of lemma is a straightforward application of the Cauchy integral formula and Montel theorem.)

Proof of Theorem. By Proposition 4.6(2) M_X is homeomorphic to the maximal ideal space of $\mathcal{O}(c_{\mathfrak{a}}X)$. It follows, e.g., from Theorem 2.9, that algebra $\mathcal{O}(c_{\mathfrak{a}}X)$ separates points of $c_{\mathfrak{a}}X$, therefore we have a continuous injection $c_{\mathfrak{a}}X \hookrightarrow M_X$ via point evaluation homomorphisms. Let us show that it is surjective.

Let $\varphi \in M_X$. We identify $\mathcal{O}(X_0)$ with $\bar{p}^* \mathcal{O}(X_0) \subset \mathcal{O}(c_{\mathfrak{a}}X)$. The restriction $\varphi|_{\mathcal{O}(X_0)}$ belongs to the maximal ideal space of $\mathcal{O}(X_0)$. Since X_0 is a Stein manifold, there exists a point $x_0 \in X_0$ such that $\varphi|_{\mathcal{O}(X_0)} = \psi_{x_0}$, where $\psi_{x_0}(u) := u(x_0)$, $u \in \mathcal{O}(X_0)$, is the evaluation homomorphism at point x_0 (see, e.g., [GR]). There exists a function $h \in \mathcal{O}(X_0)$ such that $X_0^{n-1} := \{x \in X_0 : h(x) = 0\}$ is a non-singular complex hypersurface, and $x_0 \in X_0^{n-1}$ [For]. We set $X^{n-1} := p^{-1}(X_0^{n-1})$ and $c_{\mathfrak{a}}X^{n-1} := \bar{p}^{-1}(X_0^{n-1})$. Now, if $f \in \mathcal{O}(X_0)$ is identically zero on $c_{\mathfrak{a}}X^{n-1}$, then $\varphi(f) = 0$. Indeed, by Lemma 5.9.1 function $\tilde{f} := f/\bar{p}^*h \in \mathcal{O}(c_{\mathfrak{a}}X)$, hence

$$\varphi(f) = \varphi(\tilde{f})\varphi(\bar{p}^*h) = \varphi(\tilde{f})\psi_{x_0}(h) = 0.$$

Thus, φ is well defined on the quotient algebra $\mathcal{O}(c_{\mathfrak{a}}X)/I_{c_{\mathfrak{a}}(X^{n-1})}$, where $I_{c_{\mathfrak{a}}(X^{n-1})}$ is the ideal consisting of holomorphic functions vanishing on $c_{\mathfrak{a}}X^{n-1}$. We have an isomorphism

$$\mathcal{O}(c_{\mathfrak{a}}X)/I_{c_{\mathfrak{a}}X^{n-1}} \cong \mathcal{O}(c_{\mathfrak{a}}X^{n-1}),$$

hence φ can be identified with an element of the maximal ideal space of algebra $\mathcal{O}(c_{\mathfrak{a}}X^{n-1})$.

We proceed in this way, and define sets X_0^k, X^k, $c_{\mathfrak{a}}X^k$ $(0 \leq k \leq n-1)$, obtaining that φ may be viewed as an element of the maximal ideal space of algebra $\mathcal{O}(c_{\mathfrak{a}}X^0)$, where $X_0^0 = \{x_0, x_1, \dots\}$ is a discrete set. By definition, $\mathcal{O}(c_{\mathfrak{a}}X^0) \cong \sqcup_{i\geq 0} C(\bar{p}^{-1}(x_i))$, so φ coincides with the evaluation homomorphism at a point of $\bar{p}^{-1}(x_0) \subset c_{\mathfrak{a}}X$, as needed.

Finally, it is easy to see that the topology in $c_{\mathfrak{a}}X$ is the weakest topology in which all point evaluation homomorphisms of $\mathcal{O}(c_{\mathfrak{a}}X)$ are continuous, i.e., the continuous bijection between $c_{\mathfrak{a}}X$ and M_X is a homeomorphism. □

5.10. Proofs of Theorems 2.3, 2.4, 4.10 and Proposition 4.9

Theorems 2.3 and 2.4 are the assertions (A) and (B) of Theorem 4.10.

5.10.1. **Proof of Theorem 4.10.** In what follows, all polydisks are assumed to have finite polyradii.

First, we prove part (B). We will need the following results.

Proposition 5.10.1. *Let $U := \hat{\Pi}(U_0, K)$, where $U_0 \subset X_0$ is open and biholomorphic to an open polydisk in $\mathbb{C}^n$, and $K \in \mathfrak{Q}$ (cf. (4.12)).*

The following is true:

(1) *Let $\mathcal{R}$ be an analytic sheaf over U having a free resolution of length $4N$*

$$\mathcal{O}^{k_{4N}}|_U \xrightarrow{\varphi_{4N-1}} \dots \xrightarrow{\varphi_2} \mathcal{O}^{k_2}|_U \xrightarrow{\varphi_1} \mathcal{O}^{k_1}|_U \xrightarrow{\varphi_0} \mathcal{R}|_W \longrightarrow 0. \tag{10.43}$$

If $N \geq n := \dim_{\mathbb{C}} U_0$, then the induced sequence of sections truncated to N-th term

$$\Gamma(U, \mathcal{O}^{k_N}) \xrightarrow{\varphi_{N-1}^*} \dots \xrightarrow{\varphi_2^*} \Gamma(U, \mathcal{O}^{k_2}) \xrightarrow{\varphi_1^*} \Gamma(U, \mathcal{O}^{k_1}) \xrightarrow{\varphi_0^*} \Gamma(U, \mathcal{R}) \longrightarrow 0$$

is exact.

(2) *Suppose that free resolution (10.43) exists for every N. Then $H^i(U, \mathcal{R}) = 0$, $i \geq 1$.*

Let $\mathcal{A}$ be a coherent sheaf on $c_{\mathfrak{A}}X$.

Proposition 5.10.2. *Every point $x_0 \in X_0$ has a neighbourhood U_0 such that for each $N \geq 1$ there exists a free resolution of sheaf $\mathcal{A}$ over $\bar{p}^{-1}(U_0)$ having length N (cf. Definition 2.4).*

(In other words, we may assume that the open sets W in Definition 2.4 have form $U = \bar{p}^{-1}(U_0)$, $U_0 \subset X_0$ is open.)

We prove Propositions 5.10.1 and 5.10.2 in Sections 5.10.1.2 and 5.10.1.3, respectively.

Now, let $\hat{\mathcal{A}} := \bar{p}_*\mathcal{A}$ be the direct image of sheaf $\mathcal{A}$ under projection $\bar{p} : c_{\mathfrak{A}}X \to X_0$. By definition, $\hat{\mathcal{A}}$ is a sheaf of modules over the sheaf of rings $\mathcal{O}^{C(\hat{G}_{\mathfrak{A}})}$ of germs of holomorphic functions on X_0 taking values in Banach space $C(\hat{G}_{\mathfrak{A}})$. By Propositions 5.10.2 and 5.10.1(2) every $x_0 \in X_0$ has a basis of neighbourhoods U_0 such that $H^i(U, \mathcal{A}) = 0$, $i \geq 1$, $U := \bar{p}^{-1}(U_0)$. Therefore,

$$H^i(c_{\mathfrak{A}}X, \mathcal{A}) \cong H^i(X_0, \hat{\mathcal{A}}), \quad i \geq 0 \tag{10.44}$$

(see, e.g., [Gun3, Ch. F, Cor. 6]). We have

$$\Gamma(U, \mathcal{A}) \cong \Gamma(U_0, \hat{\mathcal{A}}), \quad \Gamma(U, \mathcal{O}) \cong \Gamma(U_0, \mathcal{O}^{C(\hat{G}_{\mathfrak{A}})}).$$

It follows from Proposition 5.10.2 and Proposition 5.10.1(1) that for every $x_0 \in X_0$ and each $N \geq 1$ there exist a neighbourhood U_0 of x_0 and an exact sequence of sections

$$\Gamma(U_0, (\mathcal{O}^{C(\hat{G}_{\mathfrak{A}})})^{k_N}) \to \cdots \to \Gamma(U_0, (\mathcal{O}^{C(\hat{G}_{\mathfrak{A}})})^{k_1}) \to \Gamma(U_0, \hat{\mathcal{A}}) \to 0.$$

Then it follows that we have an exact sequence of sheaves

$$(\mathcal{O}^{C(\hat{G}_{\mathfrak{A}})})^{k_N}|_{U_0} \to \cdots \to (\mathcal{O}^{C(\hat{G}_{\mathfrak{A}})})^{k_1}|_{U_0} \to \hat{\mathcal{A}}|_{U_0} \to 0. \tag{10.45}$$

For every open set $U_0 \subset X_0$ the spaces of sections $\Gamma(U_0, \hat{\mathcal{A}})$, $\Gamma(U_0, \mathcal{O}^{C(\hat{G}_{\mathfrak{A}})})$ can be endowed with Fréchet topology, so that the homomorphisms of sections induced by sheaf homomorphisms in (10.45) are continuous; indeed, since $\Gamma(U_0, \hat{\mathcal{A}}) \cong \Gamma(U, \mathcal{A})$, $\Gamma(U_0, \mathcal{O}^{C(\hat{G}_{\mathfrak{A}})}) \cong \Gamma(U, \mathcal{O})$, this follows from Proposition 4.9 with $U = \bar{p}^{-1}(U_0)$. Hence, in the terminology of [Lt1] $\hat{\mathcal{A}}$ is a Banach coherent analytic Fréchet sheaf. Therefore, according to Theorem 2.3(iii) in [Lt1] $H^i(X_0, \hat{\mathcal{A}}) = 0$, $i \geq 1$. Isomorphism (10.44) now implies the required statement.

(C) Case (1). Due to the argument in the proof of (B), we have isomorphisms of Fréchet spaces $\Gamma(c_{\mathfrak{A}}X, \mathcal{A}) \cong \Gamma(X_0, \hat{\mathcal{A}})$, $\Gamma(\hat{Y}, \mathcal{A}) \cong \Gamma(Y_0, \hat{\mathcal{A}})$. Now the result follows from Theorem 2.3(iv) in [Lt1] applied to $\hat{\mathcal{A}}$.

Case (2). It suffices to show that the restriction map $\Gamma(\bar{p}^{-1}(Y_0), \mathcal{A}) \to \Gamma(\hat{Y}, \mathcal{A})$ has dense image, and then apply the result of case (1).

We have $\hat{Y} = \hat{\Pi}(Y_0, K)$ for some $Y_0 \Subset X_0$ open simply connected, and $K \in \mathfrak{Q}$. Since $\hat{Y} \in \mathfrak{B}$, we may use the last assertion of Proposition 4.9: it suffices to show that given a section $f \in \Gamma(\hat{Y}, \mathcal{A})$ for every $\varepsilon > 0$ and every k there exists a section $\tilde{f}_k \in \Gamma(\bar{p}^{-1}(Y_0), \mathcal{A})$ such that $\|f - \tilde{f}_k\|_{V_k} < \varepsilon$.

Without loss of generality we may identify $\hat{Y}$ with $Y_0 \times K$, and $\bar{p}^{-1}(Y_0)$ with $Y_0 \times \hat{G}_{\mathfrak{A}}$ (see Section 4.5). Then sets V_k have form $V_k = V_{0,k} \times N_k$, where each set $V_{0,k}$ is open and simply connected, and sets $N_k \in \mathfrak{Q}$ are such that $N_k \Subset N_{k+1} \Subset K$ for all k, and $K = \cup_k N_k$ (see Lemma 5.10.4(1) below). Since space $\hat{G}_{\mathfrak{A}}$ is compact and, therefore, is normal, for each k there exists a function $\rho_k \in C(\hat{G}_{\mathfrak{A}})$ such that $0 \leq \rho_k \leq 1$ on $\hat{G}_{\mathfrak{A}}$, $\rho_k \equiv 1$ on N_k, and $\rho_k \equiv 0$ on $\hat{G}_{\mathfrak{A}} \setminus \bar{N}_{k+1}$. By definition, $\Gamma(Y_0 \times K, \mathcal{A})$ is a module over $\Gamma(Y_0 \times K, \mathcal{O})$, hence we can define $\tilde{f}_k := \rho_k f \in \Gamma(Y_0 \times \hat{G}_{\mathfrak{A}}, \mathcal{A})$. Then $f - \tilde{f}_k = 0$ on $Y_0 \times N_k$, so $\|f - \tilde{f}_k\|_{V_k} = 0$. Thus, $\tilde{f}_k$ is the required approximation.

(A) Let $N \geq n$. Since sheaf $\mathcal{A}$ is coherent, there exists a neighbourhood U of x over which there is a free resolution

$$\mathcal{O}^{m_{4N}}|_U \xrightarrow{\varphi_{4N-1}} \dots \xrightarrow{\varphi_2} \mathcal{O}^{m_2}|_U \xrightarrow{\varphi_1} \mathcal{O}^{m_1}|_U \xrightarrow{\varphi_0} \mathcal{A}|_U \longrightarrow 0 \tag{10.46}$$

of length $4N$. It follows from the exactness of sequence (10.46) that there exist sections $h_1, \dots, h_{m_1} \in \Gamma(U, \mathcal{A})$ that generate ${}_x\mathcal{A}$ as an ${}_x\mathcal{O}$-module. Now, it suffices to show that there exist a neighbourhood $V \subset U$ of x, global sections $f_1, \dots, f_{m_1} \in \Gamma(c_{\mathfrak{A}}X, \mathcal{A})$ and functions $r_{ij} \in \mathcal{O}(V)$, $1 \le i, j \le m_1$, such that

$$h_i|_V = \sum_{j=1}^{m_1} r_{ij} f_j|_V, \quad 1 \le i \le m_1. \tag{10.47}$$

Without loss of generality, we may assume that $U = \hat{\Pi}(U_0, K) \in \mathfrak{B}$, where $U_0 \subset X_0$ is biholomorphic to an open polydisk in $\mathbb{C}^n$ and is holomorphically convex in X_0, and $K \in \mathfrak{Q}$. By Proposition 4.9 the topology on $\Gamma(W, \mathcal{A})$ is determined by semi-norms

$$\|h\|_{V_k} := \inf \left\{ \sup_{x \in V_k} |g(x)| : g \in \Gamma(V_k, \mathcal{O}^{m_1}),\ h = \varphi_0^*(g) \right\}, \tag{10.48}$$

where φ_0^* is the homomorphism of sections induced by φ_0 in (10.46), and open sets $V_k \in \mathfrak{B}$ are such that $V_k \Subset V_{k+1} \Subset W$ for all k, and $W = \cup_k V_k$, cf. Lemma 5.10.4(2) below; by definition, $V_k = V_{0,k} \times N_k$, where $V_{0,k} \Subset U_0$, $N_k \Subset K$ are open. Without loss of generality, we may assume that each set $V_{0,k}$ is biholomorphic to an open polydisk in $\mathbb{C}^n$ and is holomorphically convex in X_0.

Let $V := V_{k_0}$, where k_0 is chosen so that $x \in V_{k_0}$. It follows from (C) (case (2), for $\hat{Y} := U$) that for every $\varepsilon > 0$ there exist sections $f_1, \dots, f_{m_1} \in \Gamma(c_{\mathfrak{A}}X, \mathcal{A})$ such that $\|h_i - f_i\|_V < \varepsilon$. Now, by Proposition 5.10.1(1) the sequence of sections corresponding to (10.46)

$$\dots \longrightarrow \Gamma(V, \mathcal{O}^{m_1}) \xrightarrow{\varphi_0^*} \Gamma(V, \mathcal{A}) \longrightarrow 0 \tag{10.49}$$

is exact. Note that $\Gamma(V, \mathcal{O}^{m_1})$ consists of m_1-tuples of holomorphic functions on V. Let $\tilde{h}_i := (0, \dots, 1, \dots, 0)$ (1 is in the i-th position), $1 \le i \le m_1$. Without loss of generality we may assume that $h_i|_V = \varphi_0^*(\tilde{h}_i)$. Since φ_0^* is surjective, there exist functions $\tilde{f}_i \in \Gamma(V, \mathcal{O}^{m_1})$ such that $\varphi_0^*(\tilde{f}_i) = f_i|_V$. It follows from the definition of semi-norm $\|\cdot\|_V$, cf. (10.48), that functions $\tilde{f}_i$ can be chosen in such a way that

$$\sup_{x \in V} |\tilde{h}_i(x) - \tilde{f}_i(x)| < 2\varepsilon. \tag{10.50}$$

Since φ_0^* is a $\mathcal{O}(V)$-module homomorphism, the required identity (10.47) would follow once we find functions $r_{ij} \in \Gamma(V, \mathcal{O})$, $1 \le i, j \le m_1$, such that

$$\tilde{h}_i = \sum_{j=1}^{m_1} r_{ij} \tilde{f}_j, \quad 1 \le i \le m_1.$$

The latter system of linear equations (with respect to r_{ij}) can be rewritten as a matrix equation $H = FR$ with respect to $R = (r_{ij})_{i,j=1}^{m_1} \in \mathcal{O}\big(V, M_n(\mathbb{C})\big)$, where $M_n(\mathbb{C})$ denotes the set of $n \times n$ complex matrices, $H = (\tilde{h}_i)_{i=1}^{m_1} \in \mathcal{O}\big(V, GL_n(\mathbb{C})\big)$ ($\tilde{h}_i$ are the columns of H) is the identity matrix, where $GL_n(\mathbb{C}) \subset M_n(\mathbb{C})$ is the group of invertible matrices, and $F = (\tilde{f}_i)_{i=1}^{m_1} \in \mathcal{O}\big(V, M_n(\mathbb{C})\big)$ ($\tilde{f}_i$ are the columns of F). Since $\varepsilon > 0$ can be chosen arbitrarily small, in view of (10.50) we may assume that $F \in \mathcal{O}\big(V, GL_n(\mathbb{C})\big)$. Hence, we can define $R := F^{-1}H$. This completes the proof of (A).

5.10.1.1. *Auxiliary topological results.* For the proofs of Propositions 5.10.1 and 5.10.2 we will need the following results.

Let $\mathcal{L} = \{L_i\}$ be an open cover of $\hat{G}_{\mathfrak{A}}$. We define a *refinement* of $\mathcal{L}$ to be an open cover $\mathcal{L}' = \{L'_j\}$ of $\hat{G}_{\mathfrak{A}}$ such that each $L'_j \Subset L_i$ for some $i = i(j)$.

Note that since $\hat{G}_{\mathfrak{A}}$ is compact, each open cover of $\hat{G}_{\mathfrak{A}}$ has a finite subcover.

Lemma 5.10.3. *Let $\mathcal{L}$ be a finite open cover of $\hat{G}_{\mathfrak{A}}$. There exist finite refinements $\mathcal{L}^k = \{L_j^k : L_j^k \in \mathfrak{Q}\}$ of $\mathcal{L}$ having the same cardinality and such that $L_j^{k+1} \Subset L_j^k$ for all j, k.*

Proof of Lemma 5.10.3. Since $\hat{G}_{\mathfrak{A}}$ is compact, there exists a finite refinement $\mathcal{L}' = \{L_j'\}$ of $\mathcal{L} = \{L_i\}$ such that every $L_j' \Subset L_i$ for some $i = i(j)$, and functions $\{\rho_j\} \subset C(\hat{G}_{\mathfrak{A}})$ such that $\rho_j \equiv 1$ on $\bar{L}_j'$, $\rho_j \equiv 0$ on $\hat{G}_{\mathfrak{A}} \setminus L_i$. We set $L_j^k := \{\eta \in \hat{G}_{\mathfrak{A}} : \rho_j(\eta) > 1 - \frac{1}{2k}\}$, $k \geq 1$. By definition, $L_j^k \in \mathfrak{Q}$ for all j, k (cf. (4.12)). It follows that $\mathcal{L}^k := \{L_j^k\}$ are the required refinements of $\mathcal{L}$. □

Lemma 5.10.4. *Let $K \in \mathfrak{Q}$, $U_0 \subset X_0$ be open, set $U := U_0 \times K$. The following is true:*

(1) *There exist open subsets $N_k \in \mathfrak{Q}$, $1 \leq k < \infty$, such that $N_k \Subset N_{k+1} \Subset K$ for all k, and $K = \cup_k N_k$.*

(2) *There are open subsets $V_k = V_{0,k} \times N_k$, $1 \leq k < \infty$ such that $V_k \Subset V_{k+1} \Subset U$ for all k, and $U = \cup_k V_k$. Here $V_{0,k} \Subset U_0$ is open, and $N_k \in \mathfrak{Q}$, for all k.*

(3) *Let $L \in \mathfrak{Q}$ be such that $L \Subset K$. Then there exists a collection of sets $L^m \in \mathfrak{Q}$, $m \geq 1$, such that $L \Subset \cdots \Subset L^{m+1} \Subset L^m \Subset \cdots \Subset L^1 \Subset K$ for all m.*

(4) *Let $N \Subset K$, and $\{L_i\}$ be a finite collection of open subsets of K such that $N \Subset \cup_i L_i$. Then there exists a finite number of open subsets $L_j' \subset K$, $L_j' \in \mathfrak{Q}$, such that $N \Subset \cup_j L_j'$, and for each j we have $L_j' \Subset L_i$ for some $i = i(j)$.*

Proof. (1) Recall that the basis $\mathfrak{Q}$ of topology of $\hat{G}_{\mathfrak{A}}$ consists of sublevel sets of functions in $C(\hat{G}_{\mathfrak{A}})$, cf. (4.12), so $K = \{\eta \in \hat{G}_{\mathfrak{A}} : \max_{1\leq i \leq m} |h_i(\eta)) - h_i(\eta_0)| < \varepsilon\}$ for some $\eta_0 \in \hat{G}_{\mathfrak{A}}$, $h_1, \dots, h_m \in C(\hat{G}_{\mathfrak{A}})$ and $\varepsilon > 0$. Let $\mathfrak{A}'$ be the subalgebra of $C(\hat{G}_{\mathfrak{A}})$ generated by functions $h_1, \dots, h_m, \bar{h}_1, \dots, \bar{h}_m$. Since algebra $\mathfrak{A}'$ is finitely generated, the maximal ideal space $M_{\mathfrak{A}'}$ of $\mathfrak{A}'$ is a compact subset of some $\mathbb{C}^p$, and we have $\mathfrak{A}' \cong C(M_{\mathfrak{A}'})$. The map $\pi : \hat{G}_{\mathfrak{A}} \to M_{\mathfrak{A}'}$ adjoint to inclusion $\mathfrak{A}' \subset C(\hat{G}_{\mathfrak{A}})$ is proper and surjective. By definition, there exists an open subset $K' \subset M_{\mathfrak{A}'}$ such that $K = \pi^{-1}(K')$. Since $M_{\mathfrak{A}'}$ is a compact metric space (as a compact subset of $\mathbb{C}^p$), there exist open subsets $N_k' \subset M_{\mathfrak{A}'}$ such that $N_{k-1}' \Subset N_k' \Subset K'$ for all k, and $K' = \cup_k N_k'$. We define $N_k := \pi^{-1}(N_k) \in \mathfrak{Q}$. Clearly, each sets N_k' can be chosen to be a set of the form $N_k' = \{y \in M_{\mathfrak{A}'} : \max_{1\leq i \leq r_k} |f_{ik}(y) - f_{ik}(y_0)| < \varepsilon\}$ for some $y_0 \in M_{\mathfrak{A}'}$, $f_{ik} \in C(M_{\mathfrak{A}'})$ and $\varepsilon > 0$. Since $\pi^* C(M_{\mathfrak{A}'}) \subset C(\hat{G}_{\mathfrak{A}})$, we have $N_k \in \mathfrak{Q}$ (cf. (4.12)).

A similar argument yields (3).

(2) It is clear that there exists a sequence of open sets $V_{0,k}$ such that $V_{0,k} \Subset V_{0,k+1} \Subset U_0$ for all k, and $U_0 = \cup_k V_{0,k}$. We set $V_k := V_{0,k} \times N_k$.

(4) We apply Lemma 5.10.3 to the finite open cover of $\hat{G}_{\mathfrak{A}}$ consisting of the sets L_i and set $\hat{G}_{\mathfrak{A}} \setminus \bar{N}$, to obtain a finite refinement $\{L_j'\} \subset \mathfrak{Q}$ of this cover. We exclude subsets L_j' such that $L_j' \Subset \hat{G}_{\mathfrak{A}} \setminus \bar{N}$. Then $\bar{N} \subset \cup_j L_j'$ and by definition of the refinement for each j we have $L_j' \Subset L_i$ for some i, as required. □

5.10.1.2. *Proof of Proposition 5.10.1.* Let $U_0 \Subset \mathbb{C}^n$ be an open polydisk, $K \in \mathfrak{Q}$ (cf. (4.12)).

The sets $U_0 \times K$ and $\hat{\Pi}(U_0, K) \subset c_{\mathfrak{A}} X$ are biholomorphic (cf. Section 4.5). The definitions of analytic homomorphism and free resolution (of an analytic sheaf over an open subset of $c_{\mathfrak{A}} X$, cf. Section 4.7) are transferred naturally to analytic sheaves over $U_0 \times K$. Thus, it suffices to prove Proposition 5.10.1 in the assumption that analytic sheaf $\mathcal{R}$ and free resolution (10.43) are given over $U_0 \times K$.

We set $U := U_0 \times K$.

A function $f \in C(U)$ is said to be C^∞-smooth if all its derivatives with respect to variable $x \in U_0$ (in some local coordinates on U_0) are in $C(U)$. The algebra of C^∞-smooth function on U will be denoted by $C^\infty(U)$.

Let $\Lambda^{p,q}(U_0)$ be the collection of all C^∞-smooth (p,q)-forms on U_0. We define the space $\Lambda^{p,q}(U)$ of C^∞-smooth (p,q)-forms on U by the formula $\Lambda^{p,q}(U) := C^\infty(U) \otimes \Lambda^{p,q}(U_0)$. We have operator $\bar\partial : \Lambda^{p,q}(U) \to \Lambda^{p,q+1}(U)$, defined as follows: suppose that $\omega \in \Lambda^{p,q}(U)$ is given (in local coordinates on U_0) by the formula

$$\omega = \sum_{|I|=p} \sum_{|J|=q} f_{IJ} dz_I \wedge d\bar z_J, \quad f_{IJ} \in C^\infty(U),$$

where $I = (i_1, \dots, i_p)$, $J = (j_1, \dots, j_q)$, $dz_I = dz_{i_1} \wedge \dots \wedge dz_{i_p}$, $d\bar z_J = d\bar z_{j_1} \wedge \dots \wedge d\bar z_{j_q}$; then

$$\bar\partial f := \sum_{|I|=p} \sum_{|J|=q} \bar\partial f_{IJ} \wedge dz_I \wedge d\bar z_J, \tag{10.51}$$

where

$$\bar\partial f_{IJ}(z,\eta) := \sum_{j=1}^{n} \frac{\partial f_{IJ}(z,\xi)}{\partial \bar z_j} d\bar z_j, \quad z = (z_1, \dots, z_n), \quad (z,\xi) \in U = U_0 \times K.$$

A form $\omega \in \Lambda^{p,q}(U)$ is called $\bar\partial$-closed if $\bar\partial\omega = 0$.

Let $\Lambda^{p,q}$ be the sheaf of germs of C^∞-smooth (p,q)-forms on U, and $Z^{p,q} \subset \Lambda^{p,q}$ be the subsheaf of germs of $\bar\partial$-closed (p,q)-forms. Note that $Z^{0,0} = \mathcal{O}$.

Notation. We fix an open polydisk $V_0 \Subset U_0$.

Let $W_0 \subset \bar V_0$ be open in $\bar V_0$ and such that $W_0 = \bar V_0 \cap \tilde W_0$ for some product domain $\tilde W_0 = \tilde W_0^1 \times \dots \times \tilde W_0^n \Subset U_0$, where each $\tilde W_0^i \Subset \mathbb{C}$ $(1 \le i \le n)$ is simply connected and has smooth boundary (clearly, given any open neighbourhood of $\bar W_0$ in U_0, we can find such a set $\tilde W_0$ contained in this neighbourhood).

Fix a subset $W_0' \Subset W_0$ open in $\bar V_0$ and satisfying the same intersection condition as W_0.

Let $S \subset K$ be a closed subset, and let $L' \Subset L \subset S$ be open in S.

Lemma 5.10.5. *For every $\omega \in \Gamma(W_0 \times L, Z^{0,q})$ there exists $\eta \in \Gamma(\bar W_0' \times \bar L', \Lambda^{0,q-1})$ such that $\bar\partial\eta = \omega$.*

Proof. By definition, a section of sheaf $Z^{0,q}$ over $W_0 \times L$ is the restriction of a section of $Z^{0,q}$ over some open neighbourhood of $W_0 \times L$. Therefore, we may assume that L is open in K, and $\omega \in \Gamma(\tilde W_0 \times L, Z^{0,q})$ for some product domain $\tilde W_0$ as above.

Clearly, there exists a product domain $\hat W_0 \Subset \tilde W_0$ open in U_0, where $\hat W_0 = \hat W_0^1 \times \dots \times \hat W_0^n$ and each domain $\hat W_0^i \Subset \tilde W_0^i$ has smooth boundary, such that $W_0' \Subset \hat W_0$. Further, since $\hat G_{\mathfrak{A}}$ is a normal space, there exists an open set $L'' \Subset L$ such that $L' \Subset L''$.

Let $C(\bar L'')$ be the Banach space of continuous functions on $\bar L''$ endowed with sup-norm, $\Lambda^{0,q}(\tilde W_0, C(\bar L''))$ be the space of C^∞-smooth $C(\bar L'')$–valued $(0,q)$-forms on $\tilde W_0$, and

$$Z^{0,q}(\hat W_0, C(\bar L'')) \subset \Lambda^{0,q}(\hat W_0, C(\bar L''))$$

be the subspace of $\bar\partial_{C(\bar L'')}$-closed forms on $\hat W_0$. Here

$$\bar\partial_{C(\bar L'')} : \Lambda^{0,q}(\hat W_0, C(\bar L'')) \to Z^{0,q+1}(\hat W_0, C(\bar L''))$$

is the usual operator of differentiation of $C(\bar L'')$-valued forms.

It is easy to see that every form in $\Gamma(\tilde W_0 \times L, \Lambda^{0,q})$ defines a unique form in $\Lambda^{0,q}(\tilde W_0, C(\bar L''))$ and, since $\hat W_0 \times L''$ is a neighbourhood of $\bar W_0' \times \bar L'$, every form in $\Lambda^{0,q}(\tilde W_0, C(\bar L''))$ determines a unique form in $\Gamma(\bar{W}_0'\times\bar{L}',\Lambda^{0,q})$; these maps commute with the actions of operators $\bar{\partial}$ and $\bar{\partial}_{C(\bar{L}'')}$. In particular, form ω determines a form $\hat{\omega}\in Z^{0,q}(\tilde{W}_0,C(\bar{L}''))$. Note that since $\tilde{W}_0\Subset\mathbb{C}^n$ is a product domain, it is pseudoconvex. Hence W_0 admits an exhaustion by strictly pseudoconvex subdomains (see, e.g., [Kra]). Therefore, there exists a strictly pseudoconvex domain $D_0\Subset\tilde{W}_0$ such that $\hat{W}_0\Subset D_0$. We restrict form $\hat{\omega}$ to D_0 (clearly, $\hat{\omega}|_{D_0}$ is bounded), and apply Lemma 5.1.1, where we take $B:=C(\bar{L}'')$. We obtain that there exists a form $\hat{\eta}\in\Lambda^{0,q-1}(\hat{W}_0,C(\bar{L}''))$ such that $\bar{\partial}_{C(\bar{L}'')}\hat{\eta}=\hat{\omega}$ over $\hat{W}_0$. It follows that form $\eta\in\Gamma(\bar{W}_0'\times\bar{L}',\Lambda^{0,q-1})$ determined by $\hat{\eta}$ is the required one. □

DEFINITION 5.10.6. We say that a finite open cover $\mathcal{U}=\{U_\alpha\}$ of $\bar{V}_0\times S$ is *of class* (P) if the following conditions are satisfied:

(1) $U_\alpha=U_{0,l}\times L_j$, $\alpha=(l,j)$, where $\{U_{0,l}\}$ and $\{L_j\}$ are finite open covers of, respectively, $\bar{V}_0$ and S, for all α;

(2) $L_j=S\cap\tilde{L}_j$ for some $\tilde{L}_j\in\mathfrak{Q}$ such that $\tilde{L}_j\subset K$, for all j;

(3) $U_{0,l}=\bar{V}_0\cap\tilde{U}_{0,l}$ for some product domain $\tilde{U}_{0,l}=\tilde{U}_{0,l}^1\times\cdots\times\tilde{U}_{0,l}^n\Subset U_0$, where each domain $\tilde{U}_{0,l}^i\Subset\mathbb{C}$ $(1\le i\le n)$ is simply connected and has smooth boundary, for all l.

Lemma 5.10.7. *(1) Each open cover of $\bar{V}_0\times S$ has a refinement of class (P).*

(2) Each open cover of $\bar{V}_0\times S$ of class (P) has a refinement of class (P) having the same cardinality.

Proof. (1) There exists a refinement of a given open cover of $\bar{V}_0\times S$ by open sets of the form $U_{0,l}\times M_j$, $\alpha=(l,j)$, where $\{U_{0,l}\}$ and $\{M_i\}$ are finite open covers of, respectively, $\bar{V}_0$ and S. By the definition of induced topology on S, there exist open sets $\tilde{M}_i\subset K$ such that $M_i=S\cap\tilde{M}_i$. Now, we apply Lemma 5.10.4(4) to $\{\tilde{M}_i\}$ (there we take $\bar{N}:=S$) to obtain open sets $\{\tilde{L}_j\}$ such that $L_j\Subset L_i$ for some $i=i(j)$ and $\tilde{L}_j\in\mathfrak{Q}$, for all j. Finally, we set $L_j:=S\cap\tilde{L}_j$. The sets $U_{0,l}\times L_j$ form the required refinement of class (P).

(2) Follows from assertions (3) and (4) of Lemma 5.10.4. □

Let $\mathcal{U}=\{U_\alpha:=U_{0,l}\times L_j\}$ be a finite open cover of $\bar{V}_0\times S$ of class (P), and $\mathcal{U}'=\{U_\alpha':=U_{0,l}'\times L_j'\}$ be a refinement of $\mathcal{U}$ of class (P) having the same cardinality (cf. Lemma 5.10.7(2)). By definition, $\{U_{0,l}'\}$, $\{L_j'\}$ are refinements of open covers $\{U_{0,l}\}$ and $\{L_j\}$, respectively.

We have a refinement map $\iota_{\mathcal{U},\mathcal{U}'}:\mathcal{Z}^i(\mathcal{U},\mathcal{R})\to\mathcal{Z}^i(\mathcal{U}',\mathcal{R})$ (see Section 5.1.1 for notation). If no confustion arises, for a given $\sigma\in\mathcal{Z}^i(\mathcal{U},\mathcal{R})$ we denote the image $\iota_{\mathcal{U},\mathcal{U}'}(\sigma)$ again by σ.

Lemma 5.10.8. *The following is true:*

(1) *Let $\sigma\in\mathcal{Z}^i(\mathcal{U},\mathcal{O})$, $i\ge1$. Then $\sigma\in\mathcal{B}^i(\mathcal{U}',\mathcal{O})$.*

(2) *$H^i(\bar{V}_0\times S,\mathcal{O})=0$, $i\ge1$.*

Proof. (1) We will prove a more general result: if $\sigma\in\mathcal{Z}^i(\mathcal{U},Z^{0,q})$, $i\ge1$, $q\ge0$, then $\sigma\in\mathcal{B}^i(\mathcal{U}',Z^{0,q})$. In particular, taking $q=0$ we obtain assertion (1).

Let $i=1$, $\sigma_1\in\mathcal{Z}^1(\mathcal{U},Z^{0,q})$. Since $\bar{V}_0\times S$ is a paracompact space, there exist partitions of unity $\{\lambda_l\}$ and $\{\rho_j\}$ subordinate to covers $\{U_{0,l}'\}$ and $\{L_j'\}$ (C^∞-smooth and continuous, respectively). We define a 0-cocycle $\sigma_0^\infty\in C^0(\mathcal{U}',\Lambda^{0,q})$ by the formula

$$(\sigma_0^\infty)_\alpha(x,\xi):=\sum_{\beta=(l,j)}\rho_j(\xi)\lambda_l(x)(\sigma_1)_{\beta,\alpha}(x,\xi),\quad (x,\xi)\in U_\alpha',\quad \text{for all }\alpha. \tag{10.52}$$

Since $(\sigma_1)_{\alpha,\beta}=(\delta\sigma_0^\infty)_{\alpha,\beta}=(\sigma_0^\infty)_\alpha-(\sigma_0^\infty)_\beta$, and $\bar{\partial}(\sigma_1)_{\alpha,\beta}=0$, it follows that $\omega:=\bar{\partial}(\sigma_0^\infty)_\alpha$ on U_α', for all α, determines a section in $\Gamma(\bar{V}_0\times S,Z^{0,q+1})$ such that $\bar{\partial}\omega=0$. By Lemma 5.10.5

(there we take $W_0' = W_0 = \bar V_0$, and $L' = L = S$) there exists $\eta \in \Gamma(\bar V_0 \times S, \Lambda^{0,q})$ such that $\bar\partial\eta = \omega$. We define a 0-cochain $\sigma_0 \in \mathcal{C}^0(\mathcal{U}', Z^{0,q})$ by the formula $(\sigma_0)_\alpha = (\sigma_0^\infty)_\alpha - \eta$. It follows that $\sigma_1 = \delta\sigma_0$, therefore $\sigma_1 \in \mathcal{B}^1(\mathcal{U}', Z^{0,q})$.

Using Lemma 5.10.7(2) we may assume that there exists a refinement $\mathcal{U}'' = \{U''_\alpha := U''_{0,l} \times L''_j\}$ of cover $\mathcal{U}$ of class (P), having the same cardinality as $\mathcal{U}$, and such that $\mathcal{U}'$ is a refinement of $\mathcal{U}''$.

Now, let $i > 1$, assume that we have shown for all $1 \le l < i$, $q \ge 0$, that each $\sigma \in \mathcal{Z}^l(\mathcal{U}, Z^{0,q})$ belongs to $\mathcal{B}^l(\mathcal{U}'', Z^{0,q})$. For a given $\sigma_i \in \mathcal{Z}^i(\mathcal{U}, Z^{0,q})$ we define an $i-1$-cocycle $\sigma_{i-1}^\infty \in \mathcal{C}^{i-1}(\mathcal{U}'', \Lambda^{0,q})$ by the formula

$$(\sigma_{i-1}^\infty)_{\alpha_1,\dots,\alpha_i}(x,\xi) := \sum_{\beta=(l,j)} \rho_j(\xi)\lambda_l(x)(\sigma_i)_{\beta,\alpha_1,\dots,\alpha_i}(x,\xi), \quad (x,\xi) \in U''_{\alpha_1,\dots,\alpha_i}$$

for all $\alpha_1,\dots,\alpha_i$, where $U''_{\alpha_1,\dots,\alpha_i} := \cap_{r=1}^i U''_{\alpha_r} \ne \varnothing$. We have $\delta(\sigma_{i-1}^\infty) = \sigma_i$, so $\bar\partial\delta(\sigma_{i-1}^\infty) = \delta(\bar\partial\sigma_{i-1}^\infty) = 0$. Define $\mu_{i-1} := \bar\partial\sigma_{i-1}^\infty \in \mathcal{C}^{i-1}(\mathcal{U}'', Z^{0,q+1})$. Since $\delta(\mu_{i-1}) = \bar\partial\mu_{i-1} = 0$, by the induction assumption there exists an $i-2$-cochain $\mu_{i-2} \in \mathcal{C}^{i-2}(\mathcal{U}'', Z^{0,q})$ such that $\delta(\mu_{i-2}) = \mu_{i-1}$ and $\bar\partial\mu_{i-2} = 0$. Now, by Lemma 5.10.5(1) there exists an $i-2$–cochain $\eta_{i-2} \in \mathcal{C}^{i-2}(\mathcal{U}', \Lambda^{0,q})$ such that $\bar\partial\eta_{i-2} = \mu_{i-2}$. We define $\sigma_{i-1} := \sigma_{i-1}^\infty - \delta(\eta_{i-2})$. Then $\delta(\sigma_{i-1}) = \sigma_i$, so $\sigma_i \in \mathcal{B}^i(\mathcal{U}', Z^{0,q})$.

(2) By Lemma 5.10.7(1) any open cover of $\bar V_0 \times S$ has a finite refinement of class (P), hence the required result follows from (1). □

Let $\{V_k\}_{k=1}^\infty$ be the exhaustion of U by open sets obtained in Lemma 5.10.4(2). By definition, each set V_k has form $V_k = V_{0,k} \times N_k$, where $V_{0,k} \Subset U_0$, $N_k \Subset K$ are open, and $N_k \in \mathfrak{Q}$, for all k. Since U_0 is an open polydisk in $\mathbb{C}^n$, we may choose $V_{0,k}$ also to be an open polydisk, for all k.

Definition 5.10.9 (cf. [GrR]). We say that an analytic sheaf $\mathcal{R}$ on U satisfies the *Runge condition* if the following holds for every $k \ge 1$:

(a) The space of sections $\Gamma(\bar V_k, \mathcal{R})$ is endowed with a semi-norm $|\cdot|_k$ such that $\Gamma(U,\mathcal{R})|_{\bar V_k}$ is dense in $\Gamma(\bar V_k, \mathcal{R})$.

(b) There exist constants $M_k > 0$ such that for every $f \in \Gamma(\bar V_{k+1}, \mathcal{R})$ we have $|f|_{\bar V_k}|_k \le M_k|f|_{k+1}$.

(c) If $\{f_j\}$ is a Cauchy sequence in $\Gamma(\bar V_{k+1}, \mathcal{R})$, then $\{f_j|_{\bar V_k}\}$ has a limit in $\Gamma(\bar V_k, \mathcal{R})$.

(d) If $f \in \Gamma(\bar V_{k+1}, \mathcal{R})$ and $|f|_{k+1} = 0$, then $f|_{\bar V_k} = 0$.

Lemma 5.10.10 ([GrR]). *Let $\mathcal{R}$ be an analytic sheaf on U. The following is true:*

(1) *Suppose that $H^i(\bar V_k, \mathcal{R}) = 0$ for all $i \ge 1$, $k \ge 1$. Then $H^i(U, \mathcal{R}) = 0$ for all $i \ge 2$.*

(2) *If $\mathcal{R}$ satisfies the Runge condition and $H^1(\bar V_k, \mathcal{R}) = 0$ for all $k \ge 1$, then $H^1(U,\mathcal{R}) = 0$.*

Lemma 5.10.11. *The sheaf $\mathcal{O}|_U$ satisfies the Runge condition.*

Proof. For a given section $f \in \Gamma(\bar V_k, \mathcal{O})$ let us denote by $\hat f(\omega) \in \mathbb{C}$ the value of germ $f(\omega)$ at point $\omega \in \bar V_k$.

We endow each space $\Gamma(\bar V_k, \mathcal{O})$ with semi-norm $|f|_k := \sup_{\omega\in\bar V_k} |\hat f(\omega)|$. Conditions (b)–(d) are trivially satisfied. For the proof of (a), let us fix a section $f \in \Gamma(\bar V_k, \mathcal{O})$. By the definition, a section of sheaf $\mathcal{O}$ over $\bar V_k := \bar V_{0,k} \times \bar N_k$ is the restriction of a section of $\mathcal{O}$ over an open neighbourhood of $\bar V_k$. In particular, there exists an open neighbourhood $L \subset K$ of $\bar N_k$ such that section $f|_{\bar V_k}$ admits a bounded extension to $\bar V_{0,k} \times L$. Since $\hat G_{\mathfrak{A}}$ $(\supset K)$ is a normal space, there exists a function $\rho_k \in C(K)$ such that $\rho_k \equiv 1$ on $\bar{N}_k$, and $\rho_k \equiv 0$ on $K \setminus L$. We set $\tilde{f} := f\rho_k \in \Gamma(V_{0,k} \times K, \mathcal{O})$. Then function $\tilde{f}$ determines a holomorphic function $\hat{f}$ defined in a neighbourhood of $\bar{V}_{0,k}$ and with values in the Banach space $C_b(K)$ of bounded continuous functions on K endowed with sup-norm $\|\cdot\|$. We now apply the Runge-type approximation theorem for Banach-valued holomorphic functions, see [Bu2], to obtain that for every $\varepsilon > 0$ there is a function $\hat{F} \in \mathcal{O}(U_0, C_b(K))$ such that $\sup_{x\in\bar{V}_{0,k}} \|\hat{f}(x) - \hat{F}(x)\| < \varepsilon$. Then $\hat{F}$ determines a function $F \in \mathcal{O}(U)$ such that $\sup_{\omega\in\bar{V}_k} |f(\omega) - F(\omega)| < \varepsilon$, which implies (a). □

Corollary 5.10.12. *$H^i(U, \mathcal{O}) = 0$, $i \geq 1$.*

Proof. Follows from Lemmas 5.10.8(2), 5.10.10 and 5.10.11. □

Lemma 5.10.13. *Let $\mathcal{B}$, $\mathcal{R}$ be analytic sheaves on U, let $V_0 \Subset U_0$ be an open polydisk, $S \subset K$ a closed subset. Suppose that sequence*

$$\mathcal{B} \xrightarrow{\psi} \mathcal{R} \longrightarrow 0 \tag{10.53}$$

is exact. Then the sequence

$$q_*(\mathcal{B}|_{\bar{V}_0\times S}) \xrightarrow{q_*\psi} q_*(\mathcal{R}|_{\bar{V}_0\times S}) \longrightarrow 0 \tag{10.54}$$

is also exact. Here $q : \bar{V}_0 \times S \to \bar{V}_0$ is the projection on the first component, and q_* is the direct image functor.

Proof. We denote $\hat{\mathcal{B}} := q_*(\mathcal{B}|_{\bar{V}_0\times S})$, $\hat{\mathcal{R}} := q_*(\mathcal{R}|_{\bar{V}_0\times S})$, $\hat{\psi} := q_*\psi$. We have to show that $\hat{\psi}$ is surjective. Given open subsets $W_0 \subset \bar{V}_0$, $L \subset S$, we denote by $\Psi_{W_0\times L}$ the homomorphism of modules of sections $\Gamma(W_0\times L, \mathcal{B}) \to \Gamma(W_0\times L, \mathcal{R})$ induced by ψ, and by $\hat{\Psi}_{W_0}$ the homomorphism of modules of sections $\Gamma(W_0, \hat{\mathcal{B}}) \to \Gamma(W_0, \hat{\mathcal{R}})$ induced by $\hat{\psi}$. By the definition of direct image sheaf (see, e.g., [Gun3, Ch. F])

$$\Gamma(W_0 \times S, \mathcal{B}) \cong \Gamma(W_0, \hat{\mathcal{B}}), \quad \Gamma(W_0 \times S, \mathcal{R}) \cong \Gamma(W_0, \hat{\mathcal{R}}). \tag{10.55}$$

To prove exactness of (10.54) it suffices to show that for every point $x_0 \in \bar{V}_0$, a neighbourhood $W_0 \subset \bar{V}_0$ of x_0, and a section $\hat{f}_{x_0} \in \Gamma(W_0, \hat{\mathcal{R}})$, there exists a section $\hat{g}_{x_0} \in \Gamma(\tilde{W}_0, \hat{\mathcal{B}})$ over a neighbourhood $\tilde{W}_0 \subset W_0$ of x_0 such that $\hat{\Psi}_{\tilde{W}_0}(\hat{g}_{x_0}) = \hat{f}_{x_0}|_{\tilde{W}_0}$.

Let $f_{x_0} \in \Gamma(W_0 \times S, \mathcal{R})$ be the section corresponding to $\hat{f}_{x_0}$ under the second isomorphism in (10.55). By definition, a section of sheaf $\mathcal{R}$ over $W_0 \times S$ is the restriction of a section of $\mathcal{R}$ over an open neighbourhood of $W_0 \times S$. Therefore, shrinking W_0 if necessary, we obtain that f_{x_0} can be extended to a section of $\mathcal{R}$ over $W_0 \times M_1$, where $M_1 \subset K$ is an open neighbourhood of S. Since ψ is a surjective sheaf homomorphism, for each point $y \in \{x_0\} \times M_1$ there exist open sets $W_{0,y} \subset W_0$, $L_y \subset M_1$ and a section $s_y \in \Gamma(W_{0,y} \times L_y, \mathcal{B})$ such that $y \in W_{0,y} \times L_y$ and $\Psi_{W_{0,y}\times L_y}(s_y) = f_{x_0}|_{W_y\times L_y}$. Since space $\hat{G}_{\mathfrak{A}}$ ($\supset M_1$) is compact Hausdorff and, hence, is normal, there exists an open subset $M_2 \subset M_1$ such that $S \subset M_2$, and $\bar{M}_2 \subset M_1$. Since $\bar{M}_2$ is compact, there exist finitely many points $\{y_j\}_{j=1}^m \subset S$ such that $\bar{M}_2 \subset \cup_j L_{y_j}$. We set $\tilde{L}_{y_j} := \bar{M}_2 \cap L_{y_j}$, for all j. There exists a partition of unity $\{\rho_j\} \subset C(\bar{M}_2)$ subordinate to $\{\tilde{L}_{y_j}\}$. We define $\tilde{W}_0 := \cap_j W_{0,y_j}$, and set

$$g_{x_0}(z, \eta) := \sum_j \rho_j(\eta) s_{y_j}(z, \eta), \quad (z, \eta) \in \tilde{W}_0 \times S.$$

Then $g_{x_0} \in \Gamma(\tilde W_0 \times M_2, \mathcal{B})$. We have

$$\Psi_{\tilde W_0\times S}(g_{x_0}) = \sum_j \rho_j \Psi_{\tilde W_0\times \tilde L_{y_j}}(s_{y_j}) = \sum_j \rho_j f_{x_0}|_{\tilde W_0\times \tilde L_{y_j}} = f_{x_0}|_{\tilde W_0\times S}.$$

Let $\hat g_{x_0}$ denote the section in $\Gamma(\tilde W_0, \hat{\mathcal{B}})$ corresponding to g_{x_0} under the first isomorphism in (10.55). Then $\hat\Psi_{\tilde W_0}(\hat g_{x_0}) = \hat f_{x_0}|_{\tilde W_0}$, as needed. □

Definition 5.10.14. We say that an analytic sheaf $\mathcal{R}$ (on U) *admits a free resolution of length $N \geq 1$ over U* if there exists an exact sequence

$$\mathcal{F}_N|_U \xrightarrow{\varphi_{N-1}} \dots \xrightarrow{\varphi_2} \mathcal{F}_2|_U \xrightarrow{\varphi_1} \mathcal{F}_1|_U \xrightarrow{\varphi_0} \mathcal{R} \longrightarrow 0, \tag{10.56}$$

where $\mathcal{F}_i$ are free sheaves, i.e., sheaves of the form $\mathcal{O}^k$ for some $k \geq 0$ (by definition, $\mathcal{O}^0 = \{0\}$).

Lemma 5.10.15. *Let $\mathcal{R}$ be an analytic sheaf on U having a free resolution of length $3N$*

$$\mathcal{F}_{3N}|_U \xrightarrow{\varphi_{3N-1}} \dots \xrightarrow{\varphi_2} \mathcal{F}_2|_U \xrightarrow{\varphi_1} \mathcal{F}_1|_U \xrightarrow{\varphi_0} \mathcal{R} \longrightarrow 0. \tag{10.57}$$

If $N \geq n$ $(= \dim_{\mathbb{C}} U_0)$, then for each k the induced sequence of sections

$$\Gamma(\bar V_k, \mathcal{F}_N) \xrightarrow{\varphi^*_{N-1}} \dots \xrightarrow{\varphi^*_2} \Gamma(\bar V_k, \mathcal{F}_2) \xrightarrow{\varphi^*_1} \Gamma(\bar V_k, \mathcal{F}_1) \xrightarrow{\varphi^*_0} \Gamma(\bar V_k, \mathcal{R}) \longrightarrow 0 \tag{10.58}$$

is exact.

Proof. Let us fix $k \geq 1$. Let $q : \bar V_k \to \bar V_{0,k}$ be the projection, $q(x, \eta) = x$, $(x, \eta) \in V_k := V_{0,k} \times N_k$ (cf. notation before Definition 5.10.9). Let q_* denote the direct image functor, set $\hat{\mathcal{F}}_i := q_*(\mathcal{F}_i|_{\bar V_k})$, $\hat{\mathcal{R}} := q_*(\mathcal{R}|_{\bar V_k})$, $\hat\varphi_i := q_*\varphi_i$. Applying q_* to (10.57) we obtain a complex of sheaf homomorphisms

$$\hat{\mathcal{F}}_{3N} \xrightarrow{\hat\varphi_{3N-1}} \dots \xrightarrow{\hat\varphi_1} \hat{\mathcal{F}}_1 \xrightarrow{\hat\varphi_0} \hat{\mathcal{R}} \longrightarrow 0 \tag{10.59}$$

(a priori this sequence is not exact). By the definition of a direct image sheaf, the sequence of sections of (10.59) over $\bar V_{0,k}$ truncated to N-th term

$$\Gamma(\bar V_{0,k}, \hat{\mathcal{F}}_N) \to \dots \to \Gamma(\bar V_{0,k}, \hat{\mathcal{F}}_1) \to \Gamma(\bar V_{0,k}, \hat{\mathcal{R}}) \to 0 \tag{10.60}$$

coincides with sequence (10.58). Hence, the assertion would follow once we prove that sequence (10.60) is exact.

Now, exact sequence (10.57) yields a collection of short exact sequences

$$0 \longrightarrow \mathcal{R}_i|_{\bar V_k} \xrightarrow{\iota} \mathcal{F}_i|_{\bar V_k} \xrightarrow{\varphi_{i-1}} \mathcal{R}_{i-1}|_{\bar V_k} \longrightarrow 0, \quad 1 \leq i \leq 3N-1, \tag{10.61}$$

where $\mathcal{R}_i := \operatorname{Im} \varphi_i$ $(0 \leq i \leq 3N-1)$, $\mathcal{R}_0 := \mathcal{R}$, and ι stands for inclusion. We apply to (10.61) the direct image functor q_* (recall that q_* is left exact, see, e.g., [Gun3, Ch. F]) and Lemma 5.10.13 to obtain a collection of short exact sequences

$$0 \longrightarrow \mathcal{T}_i \xrightarrow{\hat\iota} \hat{\mathcal{F}}_i \xrightarrow{\hat\varphi_{i-1}} \mathcal{T}_{i-1} \longrightarrow 0, \quad 1 \leq i \leq 3N-1. \tag{10.62}$$

An argument similar to the one in the proof of Lemma 5.10.8 implies $H^l(\bar V_{0,k}, \hat{\mathcal{F}}_i) = 0$, $l \geq 1$, $k \geq 1$, $1 \leq i \leq 3N$. Hence, each short exact sequence (10.62) yields a long exact sequence of the form

$$0 \longrightarrow \Gamma(\bar V_{0,k}, \mathcal{T}_i) \longrightarrow \Gamma(\bar V_{0,k}, \hat{\mathcal{F}}_i) \longrightarrow \Gamma(\bar V_{0,k}, \mathcal{T}_{i-1}) \longrightarrow$$
$$H^1(\bar V_{0,k}, \mathcal{T}_i) \longrightarrow 0 \longrightarrow H^1(\bar V_{0,k}, \mathcal{T}_{i-1}) \longrightarrow H^2(\bar V_{0,k}, \mathcal{T}_i) \longrightarrow \dots$$

Thus, $H^m(\bar V_{0,k}, \mathcal{T}_i) \cong H^{m+1}(\bar V_{0,k}, \mathcal{T}_{i+1})$, $m \geq 1$, $1 \leq i \leq 3N-2$, and so

$$H^m(\bar V_{0,k}, \mathcal{T}_i) \cong H^{m+l+1}(\bar V_{0,k}, \mathcal{T}_{i+l+1}), \quad l \geq -1.$$

Let us take $m = 1$, $1 \leq i \leq N$, $l := 2n-2$. Then

$$H^1(\bar V_{0,k}, \mathcal{T}_i) \cong H^{2n+1}(\bar V_{0,k}, \mathcal{T}_{i+2n-1}), \quad 1 \leq i \leq N.$$

Since $N \geq n$, we have $i + 2n - 1 \leq 3N - 1$ for all $1 \leq i \leq N$, hence $\mathcal{T}_{i+2n-1}$ is well defined for all $1 \leq i \leq N$. Since the topological dimension of $\bar V_{0,k}$ is equal to $2n$, we have $H^{2n+1}(\bar V_{0,k}, \mathcal{T}_{i+2n-1}) = 0$, therefore $H^1(\bar V_{0,k}, \mathcal{T}_i) = 0$, $1 \leq i \leq N$. Therefore, we obtain collection of short exact sequences

$$0 \longrightarrow \Gamma(\bar V_{0,k}, \mathcal{T}_i) \longrightarrow \Gamma(\bar V_{0,k}, \hat{\mathcal{F}}_i) \longrightarrow \Gamma(\bar V_{0,k}, \mathcal{T}_{i-1}) \longrightarrow 0, \quad 1 \leq i \leq N,$$

which yields the exactness of sequence (10.60). The proof is complete. □

Lemma 5.10.16. *Let $\mathcal{R}$ be an analytic sheaf on U having a free resolution of length $3N$*

$$\mathcal{F}_{3N}|_U \xrightarrow{\varphi_{3N-1}} \dots \xrightarrow{\varphi_2} \mathcal{F}_2|_U \xrightarrow{\varphi_1} \mathcal{F}_1|_U \xrightarrow{\varphi_0} \mathcal{R} \longrightarrow 0. \tag{10.63}$$

If $N \geq n$, then $\mathcal{R}$ satisfies the Runge condition.

Proof. For a given section h of sheaf $\mathcal{O}^m|_U$ we denote by $\hat h(\omega) \in \mathbb{C}^m$ the value of germ $h(\omega)$ at $\omega \in U$. We have a short exact sequence

$$0 \longrightarrow \text{Ker } \varphi_0 \xrightarrow{\iota} \mathcal{F}_1|_U \xrightarrow{\varphi_0} \mathcal{R} \longrightarrow 0,$$

where ι stands for inclusion. In the proof of Lemma 5.10.15 we have shown that, under the present assumptions, for each $k \geq 1$ the sequence of sections

$$0 \longrightarrow \Gamma(\bar V_k, \text{Ker } \varphi_0) \xrightarrow{\iota^*} \Gamma(\bar V_k, \mathcal{F}_1) \xrightarrow{\varphi_0^*} \Gamma(\bar V_k, \mathcal{R}) \longrightarrow 0$$

is exact. Given a section $h \in \Gamma(\bar V_k, \mathcal{F}_1)$, we define semi-norm $|h|_k = \sup_{x \in \bar V_k} \|\hat h(x)\|$, where $\|\cdot\|$ is a Euclidean norm in $\mathbb{C}^{m_1}$, where $\mathcal{F}_1 = \mathcal{O}^{m_1}$. Now, for a section $h \in \Gamma(\bar V_k, \mathcal{R})$ we set

$$|f|_k := \inf\{|h|_k : h \in \Gamma(\bar V_k, \mathcal{F}_1), \varphi_0^*(h) = f\}. \tag{10.64}$$

We obtain a family of semi-norms $\{|\cdot|_k : k \geq 1\}$ on $\Gamma(U, \mathcal{R})$. Let us show that conditions (a)-(d) are satisfied.

(a) Let $f \in \Gamma(\bar V_k, \mathcal{R})$. There exists a section $h \in \Gamma(\bar V_k, \mathcal{F}_1)$ such that $f = \varphi_0^*(h)$. Using the same argument as in the proof of Lemma 5.10.11, we obtain that for any $\varepsilon > 0$ there exists a section $\tilde h \in \Gamma(U, \mathcal{F}_1)$ such that $|\tilde h - h|_k < \varepsilon$. We set $\tilde f := \varphi_0^*(\tilde h) \in \Gamma(U, \mathcal{R})$. By definition, $|\tilde f - f|_k < \varepsilon$, as required.

(b) Let $f \in \Gamma(\bar V_{k+1}, \mathcal{R})$. Since

$$\{h \in \Gamma(\bar V_{k+1}, \mathcal{F}_1), f = \varphi_0^*(h)\}|_{\bar V_k} \subset \{g \in \Gamma(\bar V_k, \mathcal{F}_1), f|_{\bar V_k} = \varphi_0^*(g)\},$$

and for every $h \in \Gamma(\bar V_{k+1}, \mathcal{F}_1)$ we have $|h|_k \leq |h|_{k+1}$, condition (b) is satisfied with $M_k = 1$ ($k \geq 1$) (cf. (10.64)).

(c) Let $\{f_j\}$ be a Cauchy sequence in $\Gamma(\bar V_{k+1}, \mathcal{R})$. Then there exists a Cauchy sequence $\{h_j\} \subset \Gamma(\bar V_{k+1}, \mathcal{O}^{m_1})$ such that $f_j = \varphi_0^* h_j$ for all j. Clearly, there exists a function $h \in \mathcal{O}(V_{k+1}, \mathbb{C}^{m_1}) \cap C(\bar V_{k+1}, \mathbb{C}^{m_1})$ such that

$$\sup_{\omega \in \bar V_{k+1}} |h(\omega) - \hat h_j(\omega)| \to 0 \quad \text{as } j \to \infty. \tag{10.65}$$

Then $h \in \Gamma(\bar{V}_k, O^{m_1})$, and by (10.65) $|h - h_j|_k \to 0$ $(j \to \infty)$. Now, we set $f := \varphi_0^* h \in \Gamma(\bar{V}_k, \mathcal{R})$, so by continuity $|f - f_j|_k \to 0$ $(j \to \infty)$.

(d) Let $f \in \Gamma(\bar{V}_{k+1}, \mathcal{R})$, $|f|_{k+1} = 0$. By definition, there exists a sequence of sections $h_l \in \Gamma(\bar{V}_{k+1}, \mathcal{F}_1)$ such that $f = \varphi_0^*(h_l)$ for all l, and $\sup_{\omega \in \bar{V}_{k+1}} \|\hat{h}_l(\omega)\| \to 0$ as $l \to \infty$. Let $g_l := h_1 - h_l$, $l \geq 1$. Then $g_l \in \Gamma(\bar{V}_{k+1}, \text{Ker } \varphi_0)$, and

$$\hat{g}_l(x) \to \hat{h}_1(\omega) \quad \omega \in \bar{V}_{k+1} \text{ uniformly as } l \to \infty. \tag{10.66}$$

Now, suppose that $f|_{\bar{V}_k} \neq 0$. Then $h_1|_{\bar{V}_k} \notin \Gamma(\bar{V}_k, \text{Ker } \varphi_0)$.

Consider the second fragment of the free resolution of $\mathcal{R}$,

$$0 \longrightarrow \text{Ker } \varphi_1 \xrightarrow{\iota} \mathcal{F}_2|_U \xrightarrow{\varphi_1} \text{Ker } \varphi_0 \longrightarrow 0,$$

and the corresponding sequence of sections (cf. Lemma 5.10.15)

$$0 \longrightarrow \Gamma(\bar{V}_{k+1}, \text{Ker } \varphi_1) \xrightarrow{\iota^*} \Gamma(\bar{V}_{k+1}, \mathcal{F}_2) \xrightarrow{\varphi_1^*} \Gamma(\bar{V}_{k+1}, \text{Ker } \varphi_0) \longrightarrow 0, \tag{10.67}$$

where φ_1^* is given by a matrix with entries in $\Gamma(\bar{V}_{k+1}, \mathcal{O})$. Recall that $\Gamma(\bar{V}_{k+1}, \text{Ker } \varphi_1)$ is endowed with semi-norm

$$|g|_{k+1} = \sup_{\omega \in \bar{V}_{k+1}} \|\hat{g}(\omega)\|, \quad g \in \Gamma(\bar{V}_{k+1}, \text{Ker } \varphi_1). \tag{10.68}$$

Each section in space $\Gamma(\bar{V}_{k+1}, \text{Ker } \varphi_1)$ determines a continuous function on $\bar{V}_{k+1}$ holomorphic in V_{k+1}. Let $\mathcal{A}(\bar{V}_{k+1}, \text{Ker } \varphi_1)$ be the completion of the space of these functions with respect to norm defined by (10.68). We introduce similar notation $\mathcal{A}(\bar{V}_{k+1}, \mathcal{F}_2)$, $\mathcal{A}(\bar{V}_{k+1}, \text{Ker } \varphi_0)$, for Banach spaces of holomorphic functions corresponding to two other terms in (10.67), so (10.67) yields an exact sequence of Banach spaces

$$0 \longrightarrow \mathcal{A}(\bar{V}_{k+1}, \text{Ker } \varphi_1) \xrightarrow{\iota^*} \mathcal{A}(\bar{V}_{k+1}, \mathcal{F}_2) \xrightarrow{(\varphi_0^*)'} \mathcal{A}(\bar{V}_{k+1}, \text{Ker } \varphi_0) \longrightarrow 0,$$

where $(\varphi_0^*)'$ is given by a matrix with entries in $\mathcal{A}(\bar{V}_{k+1}, \mathcal{O})$, holomorphic functions on V_{k+1} continuous on $\bar{V}_{k+1}$. It follows from (10.66) that sequence $\{g_l\}$ whose elements are viewed as functions in $\mathcal{A}(\bar{V}_{k+1}, \text{Ker } \varphi_0)$ is a Cauchy sequence and hence has a limit $g \in \mathcal{A}(\bar{V}_{k+1}, \text{Ker } \varphi_0)$. Then there exists $r \in \mathcal{A}(\bar{V}_{k+1}, \mathcal{F}_2)$ such that $g = (\varphi_1^*)'(r)$. Note that both $g|_{V_{k+1}}$, $(\varphi_1^*)'|_{V_{k+1}}$ are the sections of analytic sheaves, and in particular $g|_{\bar{V}_k} \in \Gamma(\bar{V}_k, \text{Ker } \varphi_0)$, $(\varphi_1^*)'|_{\bar{V}_k} = \varphi_1^*|_{\bar{V}_k}$. It follows that $h_1|_{\bar{V}_k} = g|_{\bar{V}_k}$, so $h_1|_{\bar{V}_k} \in \Gamma(\bar{V}_k, \text{Ker } \varphi_0)$, which is a contradiction. □

Lemma 5.10.17. *Let $\mathcal{R}$ be an analytic sheaf over U admitting a free resolution of length $4N$*

$$\mathcal{F}_{4N}|_U \xrightarrow{\varphi_{4N-1}} \dots \xrightarrow{\varphi_2} \mathcal{F}_2|_U \xrightarrow{\varphi_1} \mathcal{F}_1|_U \xrightarrow{\varphi_0} \mathcal{R} \longrightarrow 0. \tag{10.69}$$

If $N \geq n$, and for each k the sequence of sections

$$\Gamma(\bar{V}_k, \mathcal{F}_N) \xrightarrow{\varphi_{N-1}^*} \dots \xrightarrow{\varphi_2^*} \Gamma(\bar{V}_k, \mathcal{F}_2) \xrightarrow{\varphi_1^*} \Gamma(\bar{V}_k, \mathcal{F}_1) \xrightarrow{\varphi_0^*} \Gamma(\bar{V}_k, \mathcal{R}) \longrightarrow 0 \tag{10.70}$$

is exact, then the sequence of sections

$$\Gamma(U, \mathcal{F}_N) \xrightarrow{\varphi_{N-1}^*} \dots \xrightarrow{\varphi_2^*} \Gamma(U, \mathcal{F}_2) \xrightarrow{\varphi_1^*} \Gamma(U, \mathcal{F}_1) \xrightarrow{\varphi_0^*} \Gamma(U, \mathcal{R}) \longrightarrow 0 \tag{10.71}$$

is also exact.

Proof. The exact sequence (10.69) yields a collection of short exact sequences

$$0 \longrightarrow \mathcal{R}_i \xrightarrow{\iota} \mathcal{F}_i|_U \xrightarrow{\varphi_{i-1}} \mathcal{R}_{i-1} \longrightarrow 0, \quad 1 \le i \le N-1, \tag{10.72}$$

where $\mathcal{R}_i := \mathrm{Im}\ \varphi_i$ $(0 \le i \le N-1)$, $\mathcal{R}_0 := \mathcal{R}$, and ι stands for inclusion. Recall that the section functor Γ is left exact (see, e.g., [Gun3, Ch. 3]), hence we have a collection of exact sequences

$$0 \longrightarrow \Gamma(U, \mathcal{R}_i) \xrightarrow{\iota^*} \Gamma(U, \mathcal{F}_i) \xrightarrow{\varphi^*_{i-1}} \Gamma(U, \mathcal{R}_{i-1}), \quad 1 \le i \le N-1.$$

It suffices to show that φ^*_{i-1} is surjective; this would imply that (10.71) is exact.

It follows from the exactness of sequence (10.70) that for each k the sequences

$$0 \longrightarrow \Gamma(\bar{V}_k, \mathcal{R}_i) \xrightarrow{\iota^*} \Gamma(\bar{V}_k, \mathcal{F}_i) \xrightarrow{\varphi^*_{i-1}} \Gamma(\bar{V}_k, \mathcal{R}_{i-1}) \to 0, \quad 1 \le i \le N-1, \tag{10.73}$$

are exact. By Lemma 5.10.8 $H^1(\bar{V}_k, \mathcal{F}_i) = 0$, $1 \le i \le N$, for all $k \ge 1$, therefore the long exact sequence for (10.72) over $\bar{V}_k$ has the form

$$\begin{aligned} 0 \longrightarrow \Gamma(\bar{V}_k, \mathcal{R}_i) \longrightarrow \Gamma(\bar{V}_k, \mathcal{F}_i) \longrightarrow \Gamma(\bar{V}_k, \mathcal{R}_{i-1}) \to \\ H^1(\bar{V}_k, \mathcal{R}_i) \to 0 \to H^1(\bar{V}_k, \mathcal{R}_{i-1}) \to H^2(\bar{V}_k, \mathcal{R}_i) \to \dots, \quad 1 \le i \le N-1. \end{aligned}$$

Now it follows from (10.73) that $H^1(\bar{V}_k, \mathcal{R}_i) = 0$ for all $k \ge 1$, $1 \le i \le N-1$.

The long exact sequence for (10.72) over U has form

$$\begin{aligned} 0 \longrightarrow \Gamma(U, \mathcal{R}_i) \longrightarrow \Gamma(U, \mathcal{F}_i) \longrightarrow \Gamma(U, \mathcal{R}_{i-1}) \to \\ H^1(U, \mathcal{R}_i) \to H^1(U, \mathcal{F}_i) \to H^1(U, \mathcal{R}_{i-1}) \to H^2(U, \mathcal{R}_i) \to \dots, \quad 1 \le i \le N-1. \end{aligned} \tag{10.74}$$

Each sheaf $\mathcal{R}_i$, $1 \le i \le N-1$, has free resolution of length $3N$, hence by Lemma 5.10.16 it satisfies the Runge condition. It follows from Lemma 5.10.10(2) that $H^1(U, \mathcal{R}_i) = 0$ for all $1 \le i \le N-1$. We obtain from (10.74) that sequences

$$0 \longrightarrow \Gamma(U, \mathcal{R}_i) \xrightarrow{\iota^*} \Gamma(U, \mathcal{F}_i) \xrightarrow{\varphi^*_{i-1}} \Gamma(U, \mathcal{R}_{i-1}) \to 0, \quad 1 \le i \le N-1,$$

are exact, which implies the exactness of sequence (10.71). □

Proof of Proposition 5.10.1. (1) Follows from Lemmas 5.10.15 and 5.10.17.

(2) According to Lemma 5.10.16 sheaf $\mathcal{R}$ satisfies the Runge condition. Hence, by Lemma 5.10.10 we only have to show that $H^i(\bar{V}_k, \mathcal{R}) = 0$ for all $i \ge 1$ and $k \ge 1$.

Let $\mathcal{V}$ be an open cover of $\bar{V}_k := \bar{V}_{0,k} \times \bar{K}$. It suffices to show that, given an i-cocycle $\sigma \in \mathcal{Z}^i(\mathcal{V}, \mathcal{R})$ (cf. notation before Lemma 5.10.8), there exists a refinement $\mathcal{V}'$ of $\mathcal{V}$ such that the image of σ by the refinement map $\mathcal{Z}^i(\mathcal{V}, \mathcal{R}) \to \mathcal{Z}^i(\mathcal{V}', \mathcal{R})$ belongs to $\mathcal{B}^i(\mathcal{V}', \mathcal{R})$.

By Lemma 5.10.7(1) there exists a finite refinement $\mathcal{U} = \{U_\alpha\}$, $U_\alpha := U_{0,l} \times L_j$, $\alpha = (l, j)$, of cover $\mathcal{V}$ of class (P) (cf. Definition 5.10.6). Let $s = s_{\mathcal{U}}$ be the number of elements of $\mathcal{U}$, and let $N \ge \max\{n, s\}$ be the length of the free resolution of $\mathcal{R}$ over U. By the definition of open cover of class (P), a section of sheaf $\mathcal{R}$ over an element U_α of $\mathcal{U}$ admits extension to $\tilde{U}_\alpha = \tilde{U}_{0,l} \times L_j$, where $\tilde{U}_{0,l} = \tilde{U}_0^1 \times \cdots \times \tilde{U}_0^n \Subset U_0$ is a product domain such that each $\tilde{U}_0^i \subset \mathbb{C}$ $(1 \le i \le n)$ is simply connected and has smooth boundary, and $U_{0,l} = \bar{V}_{0,k} \cap \tilde{U}_{0,l}$. By part (1) of the proposition over each U_α the sequence of sections U_α corresponding to (10.43) is exact (there we can take product domain $\tilde{U}_{0,l}$ instead of polydisk U_0). Hence, we have a sequence of cochain complexes

$$\mathcal{C}^{\cdot}(\mathcal{U}, \mathcal{F}_N) \longrightarrow \dots \longrightarrow \mathcal{C}^{\cdot}(\mathcal{U}, \mathcal{F}_1) \longrightarrow \mathcal{C}^{\cdot}(\mathcal{U}, \mathcal{R}) \longrightarrow 0,$$

By Lemma 5.10.7(2) there exists a refinement $\mathcal{U}'$ of cover $\mathcal{U}$ of class (P) having the same cardinality. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\mathcal{C}^{\cdot}(\mathcal{U},\mathcal{F}_N) & \longrightarrow & \dots & \longrightarrow & \mathcal{C}^{\cdot}(\mathcal{U},\mathcal{F}_1) & \longrightarrow & \mathcal{C}^{\cdot}(\mathcal{U},\mathcal{R}) & \longrightarrow & 0\\
\downarrow & & & & \downarrow & & \downarrow & & \\
\mathcal{C}^{\cdot}(\mathcal{U}',\mathcal{F}_N) & \longrightarrow & \dots & \longrightarrow & \mathcal{C}^{\cdot}(\mathcal{U}',\mathcal{F}_1) & \longrightarrow & \mathcal{C}^{\cdot}(\mathcal{U}',\mathcal{R}) & \longrightarrow & 0
\end{array}$$

or, equivalently, the collection of commutative diagrams with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{C}^{\cdot}(\mathcal{U},\mathcal{R}_i) & \longrightarrow & \mathcal{C}^{\cdot}(\mathcal{U},\mathcal{F}_i) & \longrightarrow & \mathcal{C}^{\cdot}(\mathcal{U},\mathcal{R}_{i-1}) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{C}^{\cdot}(\mathcal{U}',\mathcal{R}_i) & \longrightarrow & \mathcal{C}^{\cdot}(\mathcal{U}',\mathcal{F}_i) & \longrightarrow & \mathcal{C}^{\cdot}(\mathcal{U}',\mathcal{R}_{i-1}) & \longrightarrow & 0
\end{array}$$

where $\mathcal{R}_i := \mathrm{Im}\ \varphi_i$ $(0 \le i \le N-1)$, $\mathcal{R}_0 := \mathcal{R}$. Each row yields a long exact sequence

$$0 \longrightarrow \Gamma(\bar{V}_k,\mathcal{R}_i) \longrightarrow \Gamma(\bar{V}_k,\mathcal{F}_i) \longrightarrow \Gamma(\bar{V}_k,\mathcal{R}_{i-1}) \longrightarrow$$
$$H^1(\mathcal{U},\mathcal{R}_i) \longrightarrow H^1(\mathcal{U},\mathcal{F}_i) \xrightarrow{\varphi^1_{i-1}} H^1(\mathcal{U},\mathcal{R}_{i-1}) \xrightarrow{\psi^2_i} H^2(\mathcal{U},\mathcal{R}_i) \longrightarrow \dots, \quad 1 \le i \le N-1$$

(and a similar one for $\mathcal{U}'$), where $H^l(\mathcal{U},\mathcal{R}_i) := \mathcal{Z}^l(\mathcal{U},\mathcal{R}_i)/\mathcal{B}^l(\mathcal{U},\mathcal{R}_i)$ are the Čech cohomology groups corresponding to cover $\mathcal{U}$. These sequences form a commutative diagram

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & H^l(\mathcal{U},\mathcal{R}_i) & \longrightarrow & H^l(\mathcal{U},\mathcal{F}_i) & \xrightarrow{\varphi^l_{i-1}} & H^l(\mathcal{U},\mathcal{R}_{i-1}) & \xrightarrow{\psi^{l+1}_i} & H^{l+1}(\mathcal{U},\mathcal{R}_i) & \longrightarrow & \dots\\
 & & \downarrow & & \downarrow{\scriptstyle \iota^l_i} & & \downarrow{\scriptstyle \gamma^l_{i-1}} & & \downarrow{\scriptstyle \gamma^{l+1}_i} & & \\
\dots & \longrightarrow & H^l(\mathcal{U}',\mathcal{R}_i) & \longrightarrow & H^l(\mathcal{U}',\mathcal{F}_i) & \xrightarrow{(\varphi^l_{i-1})'} & H^l(\mathcal{U}',\mathcal{R}_{i-1}) & \xrightarrow{(\psi^{l+1}_i)'} & H^{l+1}(\mathcal{U}',\mathcal{R}_i) & \longrightarrow & \dots
\end{array},$$

where ι^l_i, γ^l_{i-1}, γ^{l+1}_i are the corresponding refinement maps.

We have to show that, given $\sigma \in H^l(\mathcal{U},\mathcal{R})$, $l \ge 1$, there exists a refinement $\mathcal{W}$ of cover $\mathcal{U}$ such that the image of σ in $H^l(\mathcal{W},\mathcal{R})$ is zero. We construct this refinement using the following algorithm.

Suppose that there exists a non-zero $\sigma \in H^l(\mathcal{U},\mathcal{R}_{i-1})$. Let $\mathcal{U}''$ be a finite refinement of cover $\mathcal{U}'$ of class (P) having the same cardinality s as $\mathcal{U}$ and $\mathcal{U}'$ (cf. Lemma 5.10.7(2)).

We consider two cases:

(a) $\psi^{l+1}_i(\sigma) = 0$. Then there exists $\eta \in H^l(\mathcal{U},\mathcal{F}_i)$ such that $\sigma = \varphi^l_{i-1}(\eta)$. We have $\gamma^l_{i-1}(\sigma) = (\varphi^l_{i-1})'(\iota^l_i(\eta))$. By Lemma 5.10.8 $\iota^l_i(H^l(\mathcal{U},\mathcal{F}_i)) = 0$, hence the image of σ by the refinement map $\gamma^l_{i-1}(\sigma) = 0$.

(b) $\sigma' := \psi^{l+1}_i(\sigma) \ne 0$. If $\psi^{l+2}_{i+1}(\sigma') = 0$, then $\gamma^{l+1}_i(\sigma') = 0$, so by case (a) the image of σ by the refinement map $H^l(\mathcal{U}',\mathcal{R}_{i-1}) \to \mathcal{H}^l(\mathcal{U}'',\mathcal{R}_{i-1})$ is zero. If $\psi^{l+2}_{i+1}(\sigma') \ne 0$, then we apply case (b) to $\psi^{l+2}_{i+1}(\sigma')$, etc.

We apply this algorithm to $\mathcal{R}_0 = \mathcal{R}$ assuming that there exists a non-zero $\sigma \in H^l(\mathcal{U},\mathcal{R})$, $l \ge 1$. Note that case (b) can not occur after s steps: assuming the opposite, we obtain a finite refinement $\mathcal{W}$ of $\mathcal{U}$ of class (P) having the same cardinality as $\mathcal{U}$ and a non-zero element of $H^s(\mathcal{W},\mathcal{R}_i)$; however, since the cardinality of $\mathcal{U}$ is s, we have $H^s(\mathcal{W},\mathcal{R}) = 0$, which is a contradiction. Thus, after at most s steps we arrive to case (a), and hence the image of σ under the corresponding refinement map is zero. □

5.10.1.3. *Proof of Proposition 5.10.2.* The proof is based on the following lemma.

Lemma 5.10.18. *Let $U_0 \Subset \mathbb{C}^n$ be an open polydisk, and K_1, $K_2 \in \mathfrak{Q}$. Let $\mathcal{R}$ be an analytic sheaf over $U_0 \times (K_1 \cup K_2)$. Let $x_0 \in U_0$.*

Suppose that for every $N \geq 1$ sheaf $\mathcal{R}$ admits free resolutions of length N over $U_0 \times K_1$ and $U_0 \times K_2$. Then for any open subsets $L_1 \Subset K_1$, $L_2 \Subset K_2$ such that $L_i \in \mathfrak{Q}$ $(i = 1, 2)$ there exists a neighbourhood $V_0 \subset U_0$ of x_0 such that for every $N \geq 1$ sheaf $\mathcal{R}$ admits a free resolution of length N over $V_0 \times (L_1 \cup L_2)$.

We prove Lemma 5.10.18 below. Let us now complete the proof of the proposition.

Let $U_0 \Subset \mathbb{C}^n$ be an open polydisk, $x_0 \in U_0$. Since sets $\bar{p}^{-1}(U_0)$ and $U_0 \times \hat{G}_{\mathfrak{A}}$ are biholomorphic (cf. Section 4.5), it suffices to prove Proposition 5.10.2 for a coherent sheaf $\mathcal{A}$ over $U_0 \times \hat{G}_{\mathfrak{A}}$.

By definition of a coherent sheaf (cf. Definition 2.4), there exist a finite open cover of $\{x_0\} \times \hat{G}_{\mathfrak{A}}$ by sets $W_{0,i} \times L_i$, where $W_{0,i} \subset U_0$ is a neighbourhood of x_0, $\cup_i L_i = \hat{G}_{\mathfrak{A}}$, and for every $N \geq 1$ sheaf $\mathcal{A}$ admits free resolutions of length N over each $W_{0,i} \times L_i$.

By Lemma 5.10.3 there exist a collection of finite refinements

$$\mathcal{L}^k(m) = \{L_j^k : L_j^k \in \mathfrak{Q},\ 1 \leq j \leq m\}, \quad k \geq 1$$

of open cover $\{L_i\}$, such that $L_j^{k+1} \Subset L_j^k$ for all $1 \leq j \leq m$, $k \geq 1$.

Let $k = 1$. We apply Lemma 5.10.18 to sheaf $\mathcal{A}$ with $K_1 := L_{m-1}^1$, $K_2 := L_m^1$, $L_1 := L_{m-1}^2$, $L_2 := L_{m-1}^2$, $V_0 := V_{0,m} \subset \cap_i W_{0,i}$, obtaining that for each $N \geq 1$ sheaf $\mathcal{A}$ has a free resolution of length N over $V_{0,m} \times (L_{m-1}^2 \cup L_{m-1}^2)$.

We replace $\mathcal{L}^k(m)$, $k \geq 2$, with $\mathcal{L}^k(m-1) := \{L_1^k, \dots, L_{m-2}^k, L_{m-1}^k \cup L_m^k\}$, $k \geq 2$.

Now, taking $k = 2$, we apply the above procedure to cover $\mathcal{L}^2(m-1)$, obtaining that for each $N \geq 1$ sheaf $\mathcal{A}$ has a free resolution of length N over $V_{0,m-1} \times (L_{m-2}^3 \cup L_{m-1}^3 \cup L_{m-1}^3)$ for some open $V_{0,m-1} \subset V_{0,m}$, etc.

After m steps we obtain that there exists an open subset $V_{0,1} \subset \cap_j W_{0,j}$ such that for each $N \geq 1$ sheaf $\mathcal{A}$ has a free resolution over $V_{0,1} \times \hat{G}_{\mathfrak{A}}$, as required.

5.10.1.4. *Proof of Lemma 5.10.18.* We will use the following notation.

Let $M_{l \times k}(\mathbb{C})$ be the space of $l \times k$ matrices $C = (c_{ij})$ with entries $c_{ij} \in \mathbb{C}$, endowed with norm $|C| := \max\{|c_{ij}|\}_{i,j=1}^{l,k}$. We set $M_k(\mathbb{C}) := M_{k \times k}(\mathbb{C})$.

Let $GL_k(\mathbb{C}) \subset M_k(\mathbb{C})$ be the group of invertible matrices. We denote by $I = I_k \in GL_k(\mathbb{C})$ the identity matrix.

Let $U_0 \subset \mathbb{C}^n$ be an open polydisk, $K \in \mathfrak{Q}$, set $U := U_0 \times K$. The space $\mathcal{O}(U, M_k(\mathbb{C}))$ of holomorphic $M_k(\mathbb{C})$-valued functions is endowed with norm

$$\|F\|_U := \sup_{x \in U} |F(x)|, \quad F \in \mathcal{O}(U, M_k(\mathbb{C})).$$

The subset $\mathcal{O}(U, GL_k(\mathbb{C})) \subset \mathcal{O}(U, M_k(\mathbb{C}))$ of holomorphic $GL_k(\mathbb{C})$-valued maps on U has the induced topology of uniform convergence on compact subsets of U (cf. Lemma 5.10.4(2)).

The identity map $(z, \omega) \to I$, $(z, \omega) \in U$, will be denoted also by I.

Lemma 5.10.19. *Let $U' := U_0 \times K'$, $U'' := U_0 \times K''$, where K', $K'' \in \mathfrak{Q}$. Suppose that $H \in \mathcal{O}\big(U' \cap U'', GL_k(\mathbb{C})\big)$ belongs to the connected component of the identity map I in $\mathcal{O}(U' \cap U'', GL_k(\mathbb{C}))$.*

Then for any open polydisk $V_0 \Subset U_0$ and open subsets $L' \Subset K'$, $L'' \Subset K''$ there exists a function $H' \in \mathcal{O}\big(V' \cup V'', GL_k(\mathbb{C})\big)$, where $V' := V_0 \times L'$, $V'' := V_0 \times L''$, such that $H'|_{V' \cap V''} = H|_{V' \cap V''}$.

Proof. We may assume without loss of generality that polydisks V_0, U_0 are centered at the origin $0 \in \mathbb{C}^n$.

First, suppose that $\|I - H\|_{V' \cap V''} < \frac{1}{2}$, so we can define $F := \ln H \in \mathcal{O}\big(V' \cap V'', M_k(\mathbb{C})\big) \cap C\big(\bar{V}' \cap \bar{V}'', M_k(\mathbb{C})\big)$. Let us show that, after shrinking V_0, there exists a function $F' \in \mathcal{O}\big(V' \cup V'', M_k(\mathbb{C})$ such that $F'|_{V' \cap V''} = F|_{V' \cap V''}$. Indeed, we can expand the $C(\bar{L}' \cap \bar{L}'', M_k(\mathbb{C}))$-valued holomorphic function $F(z, \cdot)$ in the Taylor series around 0,

$$F(z, \eta) = \sum_{m=0}^{\infty} b_m(\eta) z^m, \quad z \in V_0, \quad \eta \in \bar{L}' \cap \bar{L}'',$$

where $b_m \in C(\bar{L}' \cap \bar{L}'', M_k(\mathbb{C}))$. Note that space $\bar{L}' \cup \bar{L}''$ is compact (and a closed subspace of a compact space $\hat{G}_{\mathfrak{A}}$), and hence is normal. Therefore, using Tietze–Urysohn extension theorem, we can extend each function b_m to a function $\tilde{b}_m \in C(\bar{L}' \cup \bar{L}'', M_k(\mathbb{C}))$ in such a way that $\sup_{\omega \in \bar{L}' \cap \bar{L}''} |b_m(\omega)| = \sup_{\omega \in \bar{L}' \cup \bar{L}''} |\tilde{b}_m(\omega)|$, and define (possibly after shrinking V_0)

$$F'(z, \omega) := \sum_{m=0}^{\infty} \tilde{b}_m(\omega) z^m, \quad z \in V_0, \quad \omega \in L' \cup L''.$$

Now, we set $H' := \exp(F') \in \mathcal{O}\big(V' \cup V'', GL_k(\mathbb{C})\big)$.

Further, let $H \in \mathcal{O}\big(U' \cap U'', GL_k(\mathbb{C})\big)$ be an arbitrary $GL_k(\mathbb{C})$-valued bounded holomorphic map belonging to the connected component of the identity map I of $\mathcal{O}\big(U' \cap U'', GL_k(\mathbb{C})\big)$.

Let us show that map $H|_{V' \cap V''}$ can be presented in the form

$$H|_{V' \cap V''} = H^1 \dots H^l, \tag{10.75}$$

where each $H^i \in \mathcal{O}(V' \cap V'', GL_k(\mathbb{C}))$, $1 \le i \le l$, satisfies

$$\|I - H^i\|_{V' \cap V''} < \frac{1}{2}. \tag{10.76}$$

Since H belongs to the connected component of the identity map I, there exists a continuous path $H_t \in \mathcal{O}\big(U' \cap U'', GL_k(\mathbb{C})\big)$ $(t \in [0, 1])$ such that $H_0 = I$, $H_1 = H$. Consider a partition $0 = t_0 < t_1 < \dots < t_l = 1$ of the unit interval $[0, 1]$, and define

$$H^i(z, \omega) = H^{-1}_{t_{i-1}}(z, \omega) H_{t_i}(z, \omega), \quad (z, \omega) \in V' \cap V'', \quad 1 \le i \le l,$$

which gives us identity (10.75). Provided that $\max_{1 \le i \le l-1} |t_{i+1} - t_i|$ is sufficiently small, inequality (10.76) holds for all $1 \le i \le l$.

Now, shrinking V_0 if necessary, we obtain according to the first case that there exist maps $H^{i\prime} \in \mathcal{O}(V' \cup V'', GL_k(\mathbb{C}))$ such that $H^{i\prime}|_{V' \cap V''} = H^i|_{V' \cap V''}$. We define $H' := H^{1\prime} \dots H^{l\prime} \in \mathcal{O}(V' \cup V'', GL_k(\mathbb{C}))$. □

Corollary 5.10.20. *In the notation of Lemma 5.10.19, for any open polydisk $V_0 \Subset U_0$ and open subsets $L' \Subset K'$, $L'' \Subset K''$ there exist functions $h' \in \mathcal{O}\big(V', GL_k(\mathbb{C})\big)$, $h'' \in \mathcal{O}\big(V'', GL_k(\mathbb{C})\big)$ such that*

$$H = h'h'', \quad \text{on } V' \cap V''.$$

Proof. Let $H' \in \mathcal{O}(V' \cup V'', GL_k(\mathbb{C}))$ be as in Lemma 5.10.19. Since $H'|_{V' \cap V''} = H|_{V' \cap V''}$, we can choose $h' := H'|_{V'}$, $h'' := I$. □

Lemma 5.10.21. *Any analytic homomorpism $\varphi : \mathcal{O}|_U^k \to \mathcal{O}|_U^l$ is determined by a holomorphic function $\Phi \in \mathcal{O}\big(U, M_{l\times k}(\mathbb{C})\big)$.*

The proof is immediate.

Definition 5.10.22 (cf. [Lem]). Let $\mathcal{R}$, $\mathcal{B}_i$, $1 \le i \le N$, be analytic sheaves over U. We say that sequence

$$\mathcal{B}_N \to \cdots \to \mathcal{B}_2 \to \mathcal{B}_1 \to \mathcal{R} \to 0 \tag{10.77}$$

is *completely exact* if for any $m \ge 1$ the sequence of sections

$$\Gamma(U, \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}^m, \mathcal{B}_N)) \to \dots \to \Gamma(U, \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}^m, \mathcal{B}_1)) \to \Gamma(U, \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}^m, \mathcal{R})) \to 0$$

or, equivalently,

$$\Gamma(U, \mathcal{B}_N^m) \to \cdots \to \Gamma(U, \mathcal{B}_1^m) \to \Gamma(U, \mathcal{R}^m) \to 0, \tag{10.78}$$

is exact.

Here $\mathcal{B}_i^m$, $\mathcal{R}^m$ stand for direct product of m copies of, respectively, $\mathcal{B}_i$, $\mathcal{R}$, with itself, and $\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}^m, \mathcal{B}_i)$, $\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}^m, \mathcal{R})$ are the sheaves of germs of analytic homomorphisms $\mathcal{O}^m \to \mathcal{B}_i$, $\mathcal{O}^m \to \mathcal{R}$, respectively.

Note that if sequence (10.78) is exact for $m = 1$, then it is exact for all $m \ge 1$.

The next two lemmas are due to [Lem].

Lemma 5.10.23. *Let $\mathcal{B}$, $\mathcal{C}$ be analytic sheaves on U. If sequence $\mathcal{B} \xrightarrow{\gamma} \mathcal{C} \to 0$ is completely exact, and $\varphi : \mathcal{O}^k|_U \to \mathcal{C}$ is an analytic homomorphism, then there is an analytic homomorphism $\psi : \mathcal{O}^k|_U \to \mathcal{B}$ such that $\varphi = \gamma\psi$.*

Proof. We can take ψ in the preimage of φ under the surjective homomorphism

$$\gamma_* : \Gamma(U, \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}^k, \mathcal{B})) \to \Gamma(U, \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}^k, \mathcal{C}))$$

(cf. Definition 5.10.22). □

Lemma 5.10.24 (Three lemma)**.** *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be analytic sheaves on U. Suppose that sequence*

$$0 \to \mathcal{A} \xrightarrow{\beta} \mathcal{B} \xrightarrow{\gamma} \mathcal{C} \to 0$$

is completely exact. If two among $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ have free resolutions of length $N + n$, where $n := \dim_{\mathbb{C}} U_0$, $N \ge n + 2$, then the third has a free resolution of length $N - n - 1$.

The proof of Lemma 5.10.24 repeats the proof of an analogous result in [Lem]. For the sake of completeness, we provide the proof below.

Proof of Lemma 5.10.18. We denote $U_1 := U_0 \times K_1$, $U_2 := U_0 \times K_2$. Let $N \ge n + 1$, consider free resolutions of $\mathcal{R}$ of length $M \ge 4N$,

$$\mathcal{O}^{k_{M,i}}|_{U_i} \longrightarrow \dots \longrightarrow \mathcal{O}^{k_{1,i}}|_{U_i} \xrightarrow{\alpha_i} \mathcal{R}|_{U_i} \longrightarrow 0, \quad i = 1, 2. \tag{10.79}$$

Consider the end portions of (10.79):

$$\mathcal{O}^{k_i}|_{U_i} \xrightarrow{\alpha_i} \mathcal{R}|_{U_i} \longrightarrow 0, \quad i = 1, 2. \tag{10.80}$$

Let $U := U_0 \times (K_1 \cup K_2)$. We denote by $\pi_i : \mathcal{O}^{k_1}|_U \oplus \mathcal{O}^{k_2}|_U \to \mathcal{O}^{k_i}|_U$, $i = 1, 2$, the natural projection homomorphisms.

First, let us show that there is an injective analytic homomorphism $H : \mathcal{O}^{k_1}|_U \oplus \mathcal{O}^{k_2}|_U \to \mathcal{O}^{k_1}|_U \oplus \mathcal{O}^{k_2}|_U$ such that $\alpha_1\pi_1 H = \alpha_2\pi_2$. By Proposition 5.10.1(1) sequence (10.79) and, hence, sequence (10.80), truncated to N-th term, are completely exact. By Lemma 5.10.23 we can factor $\alpha_1 = \alpha_2\psi$, $\alpha_2 = \alpha_1\varphi$ on $U_1 \cap U_2$ for some analytic homomorphisms $\psi : \mathcal{O}^{k_1}|_{U_1\cap U_2} \to \mathcal{O}^{k_2}|_{U_1\cap U_2}$, $\varphi : \mathcal{O}^{k_2}|_{U_1\cap U_2} \to \mathcal{O}^{k_1}|_{U_1\cap U_2}$. Now, identifying sheaf homomorphisms ψ, φ with the holomorphic matrix functions that determine them (cf. Lemma 5.10.21), we define

$$H = \begin{pmatrix} I_{k_1} & \varphi \\ 0 & I_{k_2} \end{pmatrix} \begin{pmatrix} I_{k_1} & 0 \\ \psi & I_{k_2} \end{pmatrix}^{-1} \in \mathcal{O}(U_1 \cap U_2, GL_k(\mathbb{C})),$$

where $k := k_1 + k_2$. It is immediate that $\alpha_1\pi_1 H = \alpha_2\pi_2$. The map H belongs to the connected component of the identity map in $\mathcal{O}(U_1 \cap U_2, GL_k(\mathbb{C}))$. Indeed, consider a path $H_t \in \mathcal{O}(U_1 \cap U_2, GL_k(\mathbb{C}))$ ($t \in [0,1]$),

$$H_t := \begin{pmatrix} I_{k_1} & t\varphi \\ 0 & I_{k_2} \end{pmatrix} \begin{pmatrix} I_{k_1} & 0 \\ t\psi & I_{k_2} \end{pmatrix}^{-1},$$

so that $H_0 = I_k$, $H_1 = H$.

Next, let $L_i \Subset K_i$, $L_i \in \mathfrak{Q}$ ($i = 1,2$) and $V_0 \Subset U_0$ be an open polydisk, $x_0 \in V_0$. Let $L_i^m \in \mathfrak{Q}$ ($i = 1,2$), $m \geq 1$, be the collection of open subsets of K such that $L_i \Subset L_i^{m+1} \Subset L_i^m \Subset K_i$ for all $m \geq 1$ ($i = 1,2$), obtained in Lemma 5.10.4(4).

Let $\{V_0^m\}$ be a collection of open polydisks such that $V_0 \Subset V_0^{m+1} \Subset V_0^m \Subset U_0$ for all $m \geq 1$ ($i = 1,2$).

We set $V_i^m := V_0^m \times L_i^m$ ($i = 1,2$), $m \geq 1$.

We now amalgamate the free resolutions of $\mathcal{R}$.

Let $m = 1$. By Corollary 5.10.20 there exist functions $h_i \in \mathcal{O}(V_i^1, GL_k(\mathbb{C}))$ ($i = 1,2$), such that $H = h_1 h_2$ on $V_1^1 \cap V_2^1$. Since $\alpha_1\pi_1 H = \alpha_2\pi_2$, the sheaf homomorphisms

$$\alpha_1\pi_1 h_1 : \mathcal{O}^{k_1}|_{V_1^1} \oplus \mathcal{O}^{k_2}|_{V_1^1} \to \mathcal{R}|_{V_1^1} \to 0,$$

$$\alpha_2\pi_2 h_2^{-1} : \mathcal{O}^{k_1}|_{V_2^1} \oplus \mathcal{O}^{k_2}|_{V_2^1} \to \mathcal{R}|_{V_2^1} \to 0$$

coincide over $V_1^1 \cap V_2^1$; they induce an analytic homomorphism

$$\alpha : \mathcal{O}^{k_1}|_{V_1^1\cup V_2^1} \oplus \mathcal{O}^{k_2}|_{V_1^1\cup V_2^1} \to \mathcal{R}|_{V_1^1\cup V_2^1}.$$

Let $\mathcal{R}_1 := \mathrm{Ker}\ \alpha$. The sequence

$$0 \to \mathcal{R}_1|_{V_1^1\cup V_2^1} \to \mathcal{O}^{k_1}|_{V_1^1\cup V_2^1} \oplus \mathcal{O}^{k_2}|_{V_1^1\cup V_2^1} \xrightarrow{\alpha} \mathcal{R}|_{V_1^1\cup V_2^1} \to 0$$

is completely exact over sets V_1^1 and V_2^1 since sequences (10.80) are. By Lemma 5.10.24 the analytic sheaf $\mathcal{R}_1$ has free resolutions over V_1^1 and V_2^1 (of length $4N - 2n - 1$) because two other sheaves do.

Provided that M was chosen sufficiently large, we can repeat this construction $N - 1$ times more over subsets V_1^m, V_2^m, $1 \leq m \leq N - 1$, obtaining in the end a free resolution of $\mathcal{R}$ over $V_1 \cup V_2$ having length N. Since V_0, L_1, L_2 and N were arbitrary, the required result follows. □

5.10.1.5. *Proof of Lemma 5.10.24.* We will need the following lemmas.

Lemma 5.10.25. *Let $\mathcal{A}$ be an analytic sheaf on U that admits a free resolution of length N*

$$\mathcal{F}_N|_U \xrightarrow{\varphi_{N-1}} \dots \xrightarrow{\varphi_2} \mathcal{F}_2|_U \xrightarrow{\varphi_1} \mathcal{F}_1|_U \xrightarrow{\varphi_0} \mathcal{A} \longrightarrow 0, \tag{10.81}$$

Given a completely exact sequence of analytic sheaves $\mathcal{B}_i$ on U, $0 \le i \le N$,

$$\mathcal{B}_N \xrightarrow{\beta_{N-1}} \cdots \to \mathcal{B}_2 \xrightarrow{\beta_1} \mathcal{B}_1 \xrightarrow{\beta_0} \mathcal{B}_0 \longrightarrow 0, \tag{10.82}$$

a sheaf homomorphism $\Phi_0 : \mathcal{A} \to \mathcal{B}_0$ can be extended to a homomorphism $\Phi_j : \mathcal{F}_j|_U \to \mathcal{B}_j$ $(0 \le j \le N)$ of sequences (10.56), (10.82).

Proof. The proof is by induction. We put $\varphi_{-1} := 0$ (cf. (10.81)), $\beta_{-1} := 0$. Suppose that for $0 \le j \le r$, $r \le N-1$, the homomorphisms $\Phi_j : \mathcal{F}_j|_U \to \mathcal{B}_j$ have been constructed, so that $\Phi_{j-1}\varphi_{j-1} = \beta_{j-1}\Phi_j$. If $r = N-1$, then we are done. For $r < N-1$ we have $\beta_{r-1}(\Phi_r\varphi_r) = \Phi_{r-1}\varphi_{r-1}\varphi_r = 0$. The sequence

$$\Gamma(U, \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}_{r+1}, \mathcal{B}_{r+1})) \to \cdots \to \Gamma(U, \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}_{r+1}, \mathcal{B}_0)) \to 0$$

is exact since (10.82) is completely exact (cf. Definition 5.10.22), hence there is a homomorphism $\Phi_{r+1} \in \Gamma(U, \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}_{r+1}, \mathcal{B}_{r+1}))$ such that $\Phi_r\varphi_r = \beta_r\Phi_{r+1}$ over U, as required. □

Lemma 5.10.26. *Given a free resolution (10.56) of an analytic sheaf $\mathcal{A}$ on U of length N, the sheaf* Ker $\varphi_{n-1} =$ Im φ_n *on U, $1 \le n \le N-1$, has a free resolution of length $N-n$.*

Proof. Immediate from the Definition 5.10.14 of free resolution of an analytic sheaf. □

Lemma 5.10.27. *Let $\mathcal{A}_0$ be an analytic sheaf over U. Suppose that for a given $N \ge 1$ there exists a completely exact sequence of analytic sheaves $\mathcal{A}_i$ on U, $1 \le i \le 2N+2$,*

$$\mathcal{A}_{2N+2} \overset{\alpha_{M-1}}{\to} \ldots \overset{\alpha_1}{\to} \mathcal{A}_1 \overset{\alpha_0}{\to} \mathcal{A}_0 \to 0, \tag{10.83}$$

such that sheaves $\mathcal{A}_i$, $1 \le i \le 2N+2$, have free resolutions of length $n+N$, where $n := \dim_{\mathbb{C}} U_0$. Then $\mathcal{A}_0$ has a free resolution of length N.

Proof. Let $M := 2N+2$.

(1) First, we construct a completely exact sequence of length $M-2$ of the form

$$\mathcal{B}_{M-2} \overset{\beta_{M-3}}{\to} \ldots \overset{\beta_2}{\to} \mathcal{B}_2 \overset{\beta_1}{\to} \mathcal{B}_1 \overset{\varepsilon_0}{\to} \mathcal{A}_0 \to 0, \tag{10.84}$$

where $\mathcal{B}_1 = \mathcal{O}^k|_U$ for some $k \ge 0$, a free sheaf, and $\mathcal{B}_k$, $2 \le k \le M-2$, are analytic sheaves on U having free resolutions of length $N+n-1$. Let

$$\mathcal{F}_{n+N,k} \to \cdots \to \mathcal{F}_{1,k} \overset{\omega_k}{\to} \mathcal{A}_k \to 0$$

be a free resolution of $\mathcal{A}_k$, $1 \le k \le M$. By Lemma 5.10.25 there exist analytic homomorphisms ψ_k such that the diagram

$$\begin{array}{ccccccccccc}
0 & & & & 0 & & 0 & & & & \\
\uparrow & & & & \uparrow & & \uparrow & & & & \\
\mathcal{A}_M & \xrightarrow{\alpha_{M-1}} & \ldots & \xrightarrow{\alpha_2} & \mathcal{A}_2 & \xrightarrow{\alpha_1} & \mathcal{A}_1 & \xrightarrow{\alpha_0} & \mathcal{A}_0 & \longrightarrow & 0 \\
\uparrow{\scriptstyle \omega_M} & & & & \uparrow{\scriptstyle \omega_2} & & \uparrow{\scriptstyle \omega_1} & & & & \\
\mathcal{F}_{1,M} & \xrightarrow{\psi_{M-1}} & \ldots & \xrightarrow{\psi_2} & \mathcal{F}_{1,2} & \xrightarrow{\psi_1} & \mathcal{F}_{1,1} & & & &
\end{array} \tag{10.85}$$

is commutative. Let us show that sequence

$$\mathcal{F}_{1,M} \oplus \text{Ker } \omega_{M-1} \xrightarrow{\beta_{M-1}} \ldots \xrightarrow{\beta_2} \mathcal{F}_{1,2} \oplus \text{Ker } \omega_1 \xrightarrow{\beta_1} \mathcal{F}_{1,1} \xrightarrow{\beta_0} \mathcal{A}_0 \longrightarrow 0, \tag{10.86}$$

truncated to term $\mathcal{F}_{1,M-2} \oplus \text{Ker } \omega_{M-3}$, is completely exact. Here $\beta_0 := \alpha_0\omega_1$, $\beta_1 := \psi_1 - \iota_1$, where $\iota_k : \text{Ker } \omega_k \hookrightarrow \mathcal{F}_k$ is an inclusion, and $\beta_k = (\iota_k \oplus \psi_{k-1})(\psi_k - \iota_k)$, $k \geq 2$. We apply to (10.85) and (10.86) a left exact functor $\Gamma(U, \text{Hom}_{\mathcal{O}}(\mathcal{E}, \cdot))$, where $\mathcal{E}$ is a free sheaf: let

$$A_k := \Gamma(U, \text{Hom}_{\mathcal{O}}(\mathcal{E}, \mathcal{A}_k)), \quad (0 \leq k \leq M),$$
$$F_k := \Gamma(U, \text{Hom}_{\mathcal{O}}(\mathcal{E}, \mathcal{F}_k)), \quad (1 \leq k \leq M).$$

We obtain commutative diagrams of Abelian groups

$$\begin{array}{ccccccccccc} 0 & & & & 0 & & 0 & & & & \\ \uparrow & & & & \uparrow & & \uparrow & & & & \\ A_M & \xrightarrow{a_{M-1}} & \dots & \xrightarrow{a_2} & A_2 & \xrightarrow{a_1} & A_1 & \xrightarrow{a_0} & A_0 & \longrightarrow & 0 \\ \uparrow{\scriptstyle w_M} & & & & \uparrow{\scriptstyle w_2} & & \uparrow{\scriptstyle w_1} & & & & \\ F_M & \xrightarrow{p_{M-1}} & \dots & \xrightarrow{p_2} & F_2 & \xrightarrow{p_1} & F_1 & & & & \end{array} \tag{10.87}$$

and

$$F_M \oplus \text{Ker } w_{M-1} \xrightarrow{b_{M-1}} \dots \xrightarrow{b_2} F_2 \oplus \text{Ker } w_1 \xrightarrow{b_1} F_1 \xrightarrow{b_0} A_0 \longrightarrow 0. \tag{10.88}$$

By Definition 5.10.22 the middle row of (10.87) is exact. Also, by Proposition 5.10.1(1) each w_k, $1 \leq k \leq M$, is surjective, so the columns of (10.87) are exact. Hence, we have analogous identities

$$b_0 = a_0 w_1, \quad b_1 = p_1 - i_1, \quad b_k = (i_k \oplus p_{k-1})(p_k - i_k), \tag{10.89}$$

where $i_k : \text{Ker } w_k \hookrightarrow F_k$ is an inclusion. Let us show that (10.88) is exact up to term $F_{M-2} \oplus \text{Ker } w_{M-3}$. First, note that b_0 is surjective because both a_0 and w_1 are. Second, if $\xi \in \text{Ker } b_0$, then $w_1(\xi) \in \text{Ker } a_0 = \text{Im } a_1 = \text{Im } a_1 w_2 = \text{Im } w_1 p_1$. Here we have used the fact that w_2 is surjective. Let $w_1(\xi) = w_1(p_1(\zeta))$ and $\tau := p_1(\zeta) - \xi \in \text{Ker } w_1$, so that $\xi = b_1(\zeta, \tau) \in \text{Im } b_1$. Third, if $1 \leq k \leq M-3$, and $(\xi, \eta) \in \text{Ker } b_k = \text{Ker } (p_k - i_k)$, then $\eta = p_k(\xi)$ and $0 = w_k(p_k(\xi)) = a_k(w_{k+1}(\xi))$, hence $w_{k+1}(\xi) \in \text{Im } a_{k+1} = \text{Im } a_{k+1} w_{k+2} = \text{Im } w_{k+1} p_{k+1}$. Choose ζ so that $w_{k+1}(\xi) = w_{k+1}(p_{k+1}(\zeta))$. Then $\tau := p_{k+1}(\zeta) - \xi \in \text{Ker } w_{k+1}$. We conclude that $(\xi, \eta) = b_{k+1}(\zeta, \tau) \in \text{Im } b_{k+1}$, as required.

By Lemma 5.10.26 each $\mathcal{F}_k \oplus \text{Ker } \omega_{k-1}$ has a free resolution of length $N + n - 1$. Hence, if we take

$$\mathcal{B}_1 := \mathcal{F}_1, \quad \varepsilon_0 := \beta_0, \quad \mathcal{B}_k := \mathcal{F}_k \oplus \text{Ker } \omega_{k-1}, \quad 2 \leq k \leq M-2,$$

we obtain the required completely exact sequence (10.84).

(2) Now, consider completely exact sequence (cf. (10.84))

$$\mathcal{B}_M \stackrel{\beta_{M-1}}{\to} \dots \stackrel{\beta_2}{\to} \mathcal{B}_2 \stackrel{\beta_1}{\to} \text{Ker } \varepsilon_0 \to 0.$$

We have proved that there is a completely exact sequence

$$\mathcal{D}_{M-4} \to \dots \to \mathcal{D}_3 \to \mathcal{D}_2 \stackrel{\varepsilon_1}{\to} \text{Ker } \varepsilon_0 \to 0,$$

where $\mathcal{D}_2$ is a free sheaf, and each sheaf $\mathcal{D}_k$, $3 \leq k \leq M-4$, has free resolution of length $N - n - 2$. Therefore, we have a completely exact sequence

$$\mathcal{D}_{M-4} \to \dots \to \mathcal{D}_3 \to \mathcal{D}_2 \stackrel{\varepsilon_1}{\to} \mathcal{B}_1 \to \mathcal{A}_0 \to 0.$$

Continuing in this way we obtain a free resolution of $\mathcal{A}_0$ of length N. □

Proof of Lemma 5.10.24. We can assume that $\mathcal{A} \subset \mathcal{B}$ and that β is the inclusion map.

(a) If $\mathcal{A}$ and $\mathcal{B}$ have free resolutions of length $N+n$, then Lemma 5.10.27 implies that $\mathcal{C}$ has a free resolution of length N (and, in particular, of length $N-n-1$).

Consider two remaining cases. Sheaf $\mathcal{C}$ has a free resolution of length $N+n$,

$$\mathcal{F}_{N+n} \to \cdots \to \mathcal{F}_1 \stackrel{\varphi}{\to} \mathcal{C} \to 0 \tag{10.90}$$

for some open $V_0 \subset U_0$. By Proposition 5.10.1(1) sequence (10.90) is completely exact. By Lemma 5.10.23 there is a commutative diagram

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{A} & \xrightarrow{\beta} & \mathcal{B} & \xrightarrow{\gamma} & \mathcal{C} & \longrightarrow & 0\,. \\ & & & & & \nwarrow^{\psi} & \uparrow{\scriptstyle\varphi} & & \\ & & & & & & \mathcal{F}_1 & & \end{array} \tag{10.91}$$

Let $\iota : \text{Ker } \varphi \to \mathcal{F}_1$ denote the inclusion. Let us show that the sequence

$$0 \to \text{Ker } \varphi \stackrel{\psi \oplus \iota}{\to} \mathcal{A} \oplus \mathcal{F}_1 \stackrel{\beta - \psi}{\to} \mathcal{B} \to 0 \tag{10.92}$$

is completely exact.

We apply functor $\Gamma(U, \text{Hom}_{\mathcal{O}}(\mathcal{E}, \cdot))$ to (10.91) and (10.92), where $\mathcal{E}$ is a free sheaf. We obtain diagrams of Abelian groups

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \uparrow & & \\ 0 & \longrightarrow & A & \xrightarrow{b} & B & \xrightarrow{c} & C & \longrightarrow & 0 \\ & & & & & \nwarrow^{p} & \uparrow{\scriptstyle f} & & \\ & & & & & & F_1 & & \end{array}$$

and

$$0 \to \text{Ker } f \stackrel{p \oplus i}{\to} A \oplus F_1 \stackrel{b-p}{\to} B \to 0. \tag{10.93}$$

The first diagram is commutative, its top row is exact (cf. Definition 5.10.22). By Proposition 5.10.1(1) we may assume that f is surjective. The latter sequence is a complex and is exact at Ker f. We have to check that it is exact at the next two terms. If $(\xi, \eta) \in \text{Ker } (b-p)$ then $p(\eta) = b(\xi) = \xi$, $0 = c(p(\eta)) = f(\eta)$. Thus, $\eta \in \text{Ker } f$ and $(\xi, \eta) = (p \oplus i)(\eta) \in \text{Im } (p \oplus i)$, hence (10.93) is exact in the middle term. On the other hand, if $\zeta \in B$, then with some $\eta \in F$

$$-c(\zeta) = f(\eta) = c(p(\eta)),$$

i.e.,

$$\zeta + p(\eta) = \xi \in \text{Ker } c = A.$$

Thus, $\zeta = \xi - p(\eta) \in \mathrm{Im}\,(b-p)$. We obtain that sequence (10.93) is exact, hence sequence (10.92) is completely exact.

Note that by Lemma 5.10.26 Ker φ has a free resolution of length $N+n-1$.

(b) The sheaf $\mathcal{A}\oplus\mathcal{F}_1$ has a free resolution of length $N+n-1$ over $U_0\times K$. By Lemma 5.10.27 sheaf $\mathcal{B}$ has a free resolution of length $N-1$ over $U_0\times K$ (in particular, of length $N-n-1$).

(c) We may assume that $\mathcal{B}$ has a free resolution of length $N+n-1$ over $V_0\times K$. Since Ker φ has a free resolution of length $N+n-1$, by (b) $\mathcal{A}\oplus\mathcal{F}$ has a free resolution of length $N-1$. Since sequence $0\to\mathcal{F}\to\mathcal{A}\oplus\mathcal{F}\to\mathcal{A}\to 0$ is completely exact as $\mathcal{F}$ is a free sheaf (cf. Corollary 5.10.12), we obtain by part (a) that $\mathcal{A}$ has a free resolution of length $N-n-1$. □

5.10.2. **Proof of Proposition 4.9.** The proof essentially repeats the proof of an analogous result for coherent analytic sheaves on complex manifolds, see, e.g., [GR].

First, let $\mathcal{A}$ be a coherent subsheaf of $\mathcal{O}^k$, let $U\in\mathfrak{B}$ (cf. (4.15)). By Lemma 5.10.4(2) there exist open sets $V_k\in\mathfrak{B}$ such that $V_k\Subset V_{k+1}\Subset U$ for all k, and $U=\cup_k V_k$. We endow space $\Gamma(U,\mathcal{A})$ of sections of sheaf $\mathcal{A}$ over U with the topology of uniform convergence on $\bar{V}_k$, for all k. Then $\Gamma(U,\mathcal{A})$ becomes a metrizable vector space. We have to show that space $\Gamma(U,\mathcal{A})$ is complete, i.e., it is a Fréchet space.

It is easy to see that space $\Gamma(U,\mathcal{O}^k)$ endowed with such topology is complete. Since $\mathcal{A}$ is coherent, we may assume that there exists a free resolution (2.4) of $\mathcal{A}$ over U of length $4N$, $N>n:=\dim_{\mathbb{C}}X_0$. Therefore, we have a short exact sequence

$$0\to \mathrm{Ker}\,\varphi \xrightarrow{\iota} \mathcal{O}^m|_U \xrightarrow{\varphi} \mathcal{A}|_U\to 0,$$

where ι denotes the inclusion. In the proof of Proposition 5.10.1(1) we have shown that the sequence of sections

$$0\to\Gamma(U,\mathrm{Ker}\,\varphi)\xrightarrow{\iota_*}\Gamma(U,\mathcal{O}^m)\xrightarrow{\varphi_*}\Gamma(U,\mathcal{A})\to 0 \tag{10.94}$$

is exact (cf. Lemmas 5.10.15 and 5.10.17). By our assumption $\Gamma(U,\mathcal{A})\subset\Gamma(U,\mathcal{O}^k)$. By Lemma 5.10.21 the $\Gamma(U,\mathcal{O})$-module homomorphism $\varphi_*:\Gamma(U,\mathcal{O}^m)\to\Gamma(U,\mathcal{O}^k)$ is determined by a $k\times m$ matrix with entries in $\mathcal{O}(U)$, hence it is continuous; further, ι_* is continuous. Since sequence (10.94) is exact, $\Gamma(U,\mathrm{Ker}\,\varphi)\cong\mathrm{Ker}\,\varphi_*$, hence $\Gamma(U,\mathrm{Ker}\,\varphi)$ is closed. Therefore, $\Gamma(U,\mathcal{A})$, being a quotient of a complete space by its closed subspace, is a complete space.

We note that by the open mapping theorem the topology in $\Gamma(U,\mathcal{A})$ coincides with the quotient topology determined by (10.94).

Now, let $\mathcal{A}$ be an arbitrary coherent sheaf on $c_{\mathfrak{A}}X$. Similarly, we have a free resolution (2.4) of $\mathcal{A}$ over a neighbourhood U of length $4N$, $N>n$, which yields a short exact sequence of sheaves

$$0\to \mathrm{Ker}\,\varphi \xrightarrow{\iota} \mathcal{O}^m|_U \xrightarrow{\varphi} \mathcal{A}|_U\to 0 \tag{10.95}$$

and an exact sequence of sections

$$0\to\Gamma(U,\mathrm{Ker}\,\varphi)\xrightarrow{\iota_*}\Gamma(U,\mathcal{O}^m)\xrightarrow{\varphi_*}\Gamma(U,\mathcal{A})\to 0. \tag{10.96}$$

Using Lemma 5.10.24, we obtain that Ker φ is a coherent subsheaf of $\mathcal{O}^m|_U$, so by the previous part the subspace $\Gamma(U,\mathrm{Ker}\,\varphi)\subset\Gamma(U,\mathcal{O}^m)$ is closed. We introduce in $\Gamma(U,\mathcal{A})$ the quotient topology defined by (10.96), which makes it a complete (i.e., Fréchet) space, and also implies the last assertion of the proposition concerning the semi-norms determining the topology in $\Gamma(U,\mathcal{A})$.

Let us show that thus defined topology on $\Gamma(U,\mathcal{A})$ does not depend on the choice of resolution (10.95). Suppose that there is another resolution

$$0\to \mathrm{Ker}\ \varphi' \stackrel{\iota}{\to} \mathcal{O}^{m'}|_U \stackrel{\varphi'}{\to} \mathcal{A}|_U\to 0.$$

By Lemma 5.10.25 there is a homomorphism $\psi:\mathcal{O}^m|_U\to\mathcal{O}^{m'}|_U$ such that the diagram of exact sequences of sheaves

$$\begin{array}{ccccc} \mathcal{O}^m|_U & \stackrel{\varphi}{\longrightarrow} & \mathcal{A}|_U & \longrightarrow & 0\\ \psi\downarrow & & \downarrow\lambda & & \\ \mathcal{O}^{m'}|_U & \stackrel{\varphi'}{\longrightarrow} & \mathcal{A}|_U & \longrightarrow & 0 \end{array}$$

is commutative. Therefore, we have a commutative diagram

$$\begin{array}{ccccc} \Gamma(U,\mathcal{O}^m) & \stackrel{\varphi_*}{\longrightarrow} & \Gamma(U,\mathcal{A}) & \longrightarrow & 0\\ \psi_*\downarrow & & \downarrow\lambda_* & & \\ \Gamma(U,\mathcal{O}^{m'}) & \stackrel{\varphi'_*}{\longrightarrow} & \Gamma(U,\mathcal{A}) & \longrightarrow & 0 \end{array}$$

of exact sequences of sections. By our construction φ_*, φ'_* are continuous and surjective, ψ_* is continuous as a homomorphism of sections of free sheaves. By the open mapping theorem the preimage of an open set by $\lambda_*^{-1}=\varphi_*(\psi_*)^{-1}(\varphi'_*)^{-1}$ is open, so λ_* is continuous, hence it is a homeomorphism.

Finally, let $\gamma:\mathcal{A}\to\mathcal{B}$ be an analytic homomorphism. Let us show that γ is continuous. Analogously to the previous part, applying Lemma 5.10.25, we obtain a commutative diagram of exact sequences of sheaves, which yields a commutative diagram of exact sequences

$$\begin{array}{ccccc} \Gamma(U,\mathcal{O}^m) & \stackrel{\varphi_*}{\longrightarrow} & \Gamma(U,\mathcal{A}) & \longrightarrow & 0\\ \psi_*\downarrow & & \downarrow\gamma_* & & \\ \Gamma(U,\mathcal{O}^{m'})|_U & \stackrel{\varphi'_*}{\longrightarrow} & \Gamma(U,\mathcal{B}) & \longrightarrow & 0 \end{array}.$$

As before, the continuity of γ_* can be deduced from the continuity of the other homomorphisms, as before. This completes the proof of the proposition.

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