Algebraic stability

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It is possible to define the action of a map on cohomology group even if a map is not regular. Algebraic stability is the property a meromorphic map that makes it especially nice for dynamical study.

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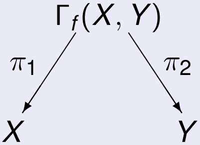
Definition

Bimeromorphic map (correspondence) $f : \mathbb{X} \to Y$ is defined by its graph (closure) $\Gamma_f \subset X \times Y$, an irreducible subvariety. Let $\pi_1 : \Gamma_f \to X$, $\pi_2 : \Gamma_f \to Y$ be projections.



We require that π_1^{-1}, π_2^{-1} are well-defined everywhere except for finitely many points.

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- Indeterminacy set $I(\widetilde{H}_{a,c}) = \{[0 : 1 : 0]\}$.
- Exceptional set $E(\widetilde{H}_{a,c}) = \{z = 0\}$.
  $\widetilde{H}_{a,c}(\{z = 0\}) = [1 : 0 : 0] = E(\widetilde{H}_{a,c}^{-1})$

Action on cohomology.

Despite the fact that map is not regular there is a natural way to define it's action on cohomology $f^* : H^*(X) \to H^*(Y)$.
Let $\tilde{\Gamma}_f$ be minimal disingularization of Γ_f. Then one can lift π_1, π_2 to $\tilde{\Gamma}_f$ to be $\tilde{\pi}_1, \tilde{\pi}_2$

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Action f^* on form α is defined

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Note that $f^* d = d f^*$, therefore, it descends to **action on cohomology** group.

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$H^{1,1}(\mathbb{C}P^2) = \text{Pic}(\mathbb{C}P^2) = \mathbb{Z}$
It is generated by the hyperplane section L. The class of curve in $\mathbb{C}P^2$ is determined by its degree.

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The action f^* on $\mathrm{Pic}(\mathbb{C}P^2)$ is multiplication by d, where d is algebraic degree of f written in homogeneous coordinates.

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This quantity dominates topological entropy $h_{\text{top}}(f) \leq \log \lambda_1(f)$, and the equality is conjectured [F].

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If $(f^n)^* = (f^*)^n$, then λ_1 is equal to the *spectral radius* $\rho(f^*)$
$\lambda_(f)$ can be read from the first iterate of f.

Examples of algebraically stable and unstable maps.

One can check that $H_{a,c}^k$ has first coordinate $x^{2^k} + ...$, therefore algebraic degree of $\widetilde{H}_{a,c}^k$ is 2^k and it is **algebraically stable.**

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  $L = (1 : y : y + 1)$
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- $\tilde{F}_C^2 = Id$.
- Thus, $(\tilde{F}_C^*)^2 \neq (\tilde{F}_C^2)^*$.
  $\tilde{F}_C$ is **NOT algebraically stable.**

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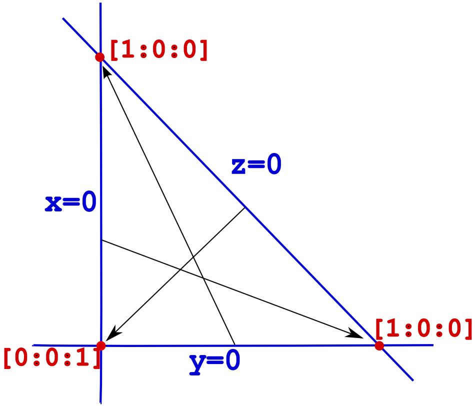

Characterization of algebraic stability.

Theorem

Let X be a compact complex surface and $f : X \circlearrowleft$ is a bimeromorphic map and f^* the induced action on $H^{1,1}(X)$

1. $(f^*)^n = (f^n)^*$
2. $f^n(E) \not\subset I(f)$ *for any exceptional curve E.*

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Let's check the criterion for **Hénon maps.**

$E(\widetilde{H}_{a,c}) = \{z = 0\}$
$\widetilde{H}_{a,c}(E) = (1 : 0 : 0) \subset \{z = 0\}$
$\widetilde{H}^k_{a,c}(E) = (1 : 0 : 0) \notin (0 : 1 : 0) = I(\widetilde{H}_{a,c})$

Algebraically stable model.

Theorem

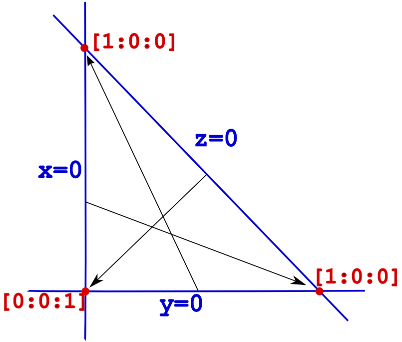
For any bimeromorphic map $f : X \circlearrowleft$ *of the complex compact surface* X *there exists a complex surface* $\tilde{X}$ *and* $\pi : \tilde{X} \to X$, *which is a finite composition of blow-downs, such that the diagram commutes*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{X} \\ \downarrow \pi & & \downarrow \pi \\ X & \xrightarrow{f} & X \end{array}$$

and $\tilde{f}$ *is algebraically stable.*

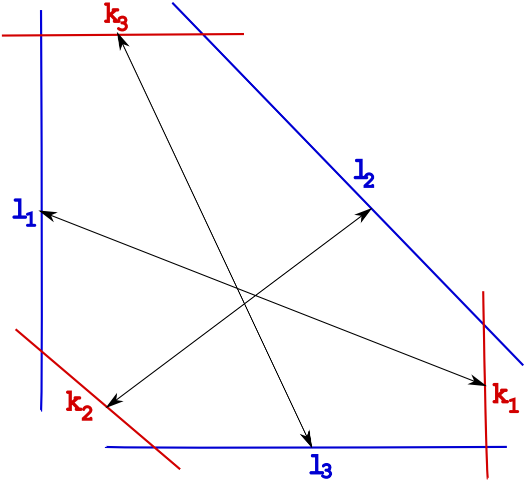
Cremona Transform

$\tilde{F}_C(x=0) = (1:0:0) \in I(\tilde{F}_C)$, $\tilde{F}_C(y=0) = (0:1:0) \in I(\tilde{F}_C)$, $\tilde{F}_C(z=0) = (0:0:1) \in I(\tilde{F}_C)$



Algebraically stable model for Cremona Transform

$\pi(l_1) = \{x = 0\}$, $\pi(l_2) = \{z = 0\}$, $\pi(l_3) = \{y = 0\}$
$\pi(k_1) = (1 : 0 : 0)$, $\pi(k_2) = (0 : 0 : 1)$, $\pi(k_3) = (0 : 1 : 0)$



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