Dynamical Foliations.

by

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Abstract

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This thesis is devoted to the study of foliations that come from dynamical systems.

In the first part we study foliations of Stein manifolds locally given by vector fields. The leaves of such foliations are Riemann surfaces. We prove that for a generic foliation all leaves except for not more than a countable number are homeomorphic to disks, the rest are homeomorphic to cylinders. We also prove that a generic foliation is complex Kupka-Smale.

In the second part of the thesis we study complex Hénon maps. The sets of points U^+ and U^- that have unbounded forward and backwards orbits correspondingly, is naturally endowed with holomorphic foliations $\mathcal{F}^+$ and $\mathcal{F}^-$. We describe the critical locus – the set of tangencies between these foliations – for Hénon maps that are small perturbations of quadratic polynomials with disconnected Julia set.

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Contents

Chapter 1

Generic properties of holomorphic foliations on Stein manifolds

1.1 Introduction

This chapter is devoted to the study of 1-dimensional (singular) holomorphic foliations of Stein manifolds. The foliations of $\mathbb{C}^n$ serve as the main example for us.

We restrict ourselves to the foliations with singular locus of codimension 2. Such foliations are locally determined by vector fields [Il72]. In case of $\mathbb{C}^n$ such foliations are globally determined by vector fields.

We study generic properties of such foliations. The genericity here is understood as follows: the space of holomorphic foliations can be naturally equipped with (Baire) topology of uniform convergence on nonsingular compacts sets. We provide the definition and a thorough discussion of the topology in Subsection 1.2.4. We call a foliation generic if it belongs to a residual set – an intersection of countably many open everywhere dense sets.

We describe the topology of leaves for a generic foliation. Moreover, we prove that a generic foliation is complex Kupka-Smale. Main results are given in the following two theorems:

Theorem 1.1.1. *For a generic* 1*-dimensional singular holomorphic foliation of a Stein manifold* X *all leaves except for not more than a countable number are topological disks, the rest are topological cylinders.*

Theorem 1.1.2. *A generic* 1*-dimensional singular holomorphic foliation of* X *is complex Kupka-Smale.*

Complex Kupka-Smale property is a an analog of real Kupka-Smale property.

Definition 1.1.1. A foliation is called *complex Kupka-Smale* if

1. all its singular points are complex hyperbolic;
2. all complex cycles are hyperbolic;
3. strongly invariant manifolds of different singular points intersect transversally.

The above definition was suggested by Marc Chaperon in [Ch04]. In this preprint he studies holomorphic 1-dimensional singular foliations of Stein manifolds. He shows that the property (1) holds for generic foliations. He also gives the proof of the property (3) for generic foliations of $\mathbb{C}^n$ and states the result for generic foliations of Stein manifolds. To make the paper self-contained we give a proof of (3). Moreover, we show that using our technique we prove transversality results for strongly invariant manifolds of the same singular point.

We prove the following theorem:

Theorem 1.1.3. *There exists a residual set in the space of holomorphic* 1*-dimensional singular foliations such that*

1. *all singular points are complex hyperbolic.*

2. *Let* a_1 *be a complex hyperbolic singular point of the foliation. Let* M_1 *and* M_2 *be strongly invariant manifolds of the point* a_1*, such that* $M_1^{loc} \cap M_2^{loc} = a_1$*. Then* M_1 *and* M_2 *intersect transversally everywhere.*

The notions of complex hyperbolicity, strongly invariant manifolds and complex cycles are reviewed in Section 1.2.

We establish generic properties of foliations by constructing perturbations that eliminate degeneracies. To prove Theorem 1.1.1 one needs to eliminate all degeneracies from the following list:

1. non-isolated cycles;
2. two cycles that belong to the same leaf of the foliation and are not multiples of the same cycle in the homology group of the leaf;
3. saddle connections;
4. cycles on a separatrix that are not multiples of the cycle around the critical point.

The definitions of a separatrix and a saddle connection are reviewed in Section 1.2.

There are not more than a countable number of isolated cycles by Landis-Petrovskii's Lemma [LP55]. In their paper the statement is proved for the foliations of $\mathbb{C}^2$. To make the thesis self-contained we provide the proof of the lemma for the foliations of Stein manifolds in 1.6.1. Therefore, once all non-isolated cycles are eliminated, all leaves except for at most countably many are contractible.

In the smooth category one can destroy a degeneracy of the foliation locally. Say, one can destroy a homoclinic loop by changing the foliation only in a flow-box around a point on the loop:

In the holomorphic category one cannot perturb the foliation in the flow-box without changing the foliation globally.

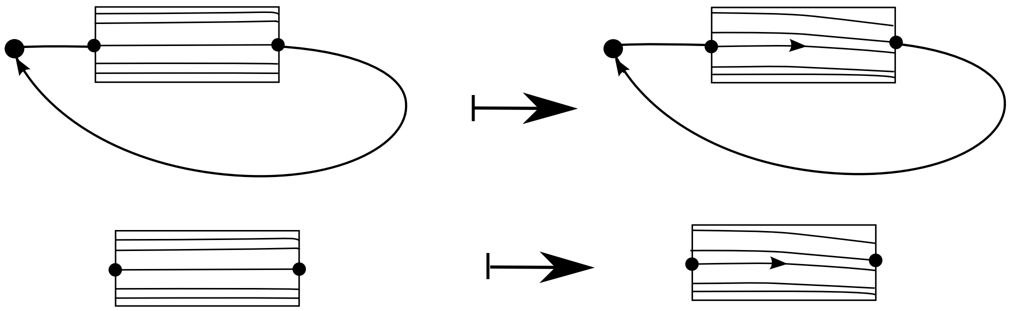


Figure 1.1: A homoclinic separatrix connection

We solve this problem in two steps. First, we construct a family of foliations in a neighborhood of a degenerate object (i.e. a non-hyperbolic cycle; a non-trivial pair of cycles on the leaf; a pair of paths on strongly invariant manifolds that connect two singular points of foliation with a point of non-transversal intersection of these manifolds). Second, we approximate the foliation in the neighborhood by a global one, using the approximation theory on Stein manifolds and the description of the holomorphic hull of a curve in $\mathbb{C}^n$. The second step can only be carried if the degenerate object is holomorphically convex. So we show that we can restrict attention to holomorphically convex degenerate objects.

When we eliminate a degeneracy, i.e. a complex cycle, we do not have control over the foliation outside the neighborhood of the degeneracy. Therefore, it might happen that eliminating one degeneracy we create many other in different places. This problem is solved indirectly. We find a countable number of places where the degeneracies can be located. For each such location we prove that the complement to the set of foliations, which have a degeneracy at this particular location, is open and everywhere dense. (The proof substantially uses the holomorphic nature of the foliations.) Then we intersect these sets and get the residual set of foliations without degeneracies. We describe this strategy in detail in Section 1.6. This strategy was previously used in [F06] and [GKK].

1.1.1 Outline of the chapter.

In Section 1.2 we provide the necessary background.

We give an overview of the history of the subject in Section 1.3.

As we pointed out there are no local perturbations allowed in the holomorphic category, since one cannot change the foliation just in the flow-box. For holomorphically convex degenerate objects we construct a neighborhood and a family of foliations that eliminate the degeneracy in the neighborhood of the degenerate object.

We give a review of results on the holomorphic hulls of a collection of curves in Section 1.5. We apply them to give geometric conditions for the degenerate object to be holomorphically convex. We also review the relevant results from the approximation theory on Stein manifolds and apply them to pass from a local foliation in a neighborhood of a degenerate object to a global one.

Section 1.6 is devoted to the simultaneous elimination of degeneracies. In this section we construct countably many holonomy maps that catch all degeneracies.

1.2 Background information.

1.2.1 Stein manifolds.

In this subsection we state the well-known facts about Stein manifolds. For the proofs and further discussion, consult [H90].

For smooth manifolds there is Whitney Embedding Theorem, stating that any smooth m-dimensional manifold can be smoothly embedded into Euclidean $2m$-space. For complex holomorphic manifolds the situation is different. There are complex manifolds that cannot be holomorphically embedded as submanifolds to $\mathbb{C}^n$. Moreover, there are ones that do not admit any global holomorphic functions, except for constants.

By Maximum Modulus Theorem and Liouville's Theorem compact manifolds do not admit any nonconstant global holomorphic functions.

Informally speaking, Stein manifolds are the ones which do have an ample supply of holomorphic functions.

We start our discussion of Stein manifolds with the definition of the holomorphic hull. This notion plays an important role in the theory.

Definition 1.2.1. Let K be a compact subset of a complex manifold X, the $\mathcal{O}(X)$-*hull* of K is the set

$$h_X(K) = \{u: \ |f(u)| \leq \max\{f(x)|x \in K\}\text{for all} f \in \mathcal{O}(X)\},$$

where $\mathcal{O}(X)$ are holomorphic functions on X.

Note 1.2.1. *We also call $\mathcal{O}(X)$-hull, the holomorphic hull, when it is clear from the context what the ambient manifold is. The notation $h(K)$ is used in that case.*

Note 1.2.2. *The holomorphic hull is a reasonable notion, only if the manifold has an ample supply of holomorphic functions. For instance, it is an important notion for the compact subsets of $\mathbb{C}^n$.*

Example 1.2.1. *The holomorphic hull of the curve $\{|z| = 1\} \subset \mathbb{C}$ is $\{|z| \leq 1\}$, i.e. the curve together with interior in $\mathbb{C}$.*

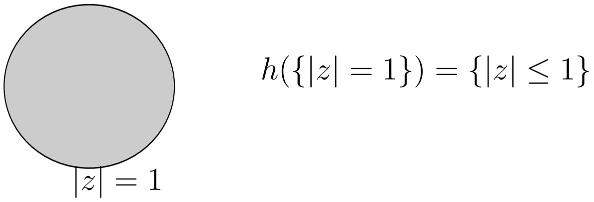


Figure 1.2: The holomorphic hull of the curve $\{|z| = 1\}$ in $\mathbb{C}$

Proof. By Maximum Modulus Principle, the points z such that $\{|z| \leq 1\}$ belong to the holomorphic hull.

Take a point z_0 so that $|z_0| > 1$. By considering the global holomorphic function z we see that this point does not belong to the holomorphic hull. □

Example 1.2.2. *Consider a curve* γ

$$\{(z, w) \in \mathbb{C}^2 | \ |z| = 1, z = \bar{w}\}.$$

Then $h_{\mathbb{C}^2}(\gamma) = \gamma$.

Proof. The function $f(z, w) = zw - 1$ is equal to zero on γ. Therefore, $h(\gamma) \subset \{f = 0\}$. Take a point $(z_0, w_0) \in \mathbb{C}^2$.

- If $|z_0| > 1$. Then

  $$|z_0| > \max\{|z| : \ (z, w) \in \gamma\}$$

  Function z is a global holomorphic function. Therefore, the point (z_0, w_0) does not belong to $h(\gamma)$.

- If $|z_0| < 1$, then $|w_0| > 1$.

  $$|w_0| > \max\{|w| : \ (z, w) \in \gamma\}$$

  Therefore, the point (z_0, w_0) does not belong to $h(\gamma)$.

- If $|z_0| = 1$, then $z_0 = \bar{w}_0$. So $(z_0, w_0) \in \gamma$.

Thus, $h(\gamma) = \gamma$ □

Definition 1.2.2. Complex analytic manifold X of dimension n is said to be *a Stein manifold* if

1. for every compact set K its holomorphic hull $h(K)$ is also compact;

2. If z_1 and z_2 are two different points of X, then $f(z_1) \neq f(z_2)$ for some $f \in \mathcal{O}(X)$;
3. For every $z \in X$, one can find n functions $f_1, \ldots, f_n \in \mathcal{O}(X)$ which form a coordinate system at z.

Fact 1.2.1. *$\mathbb{C}^n$ is a Stein manifold.*

Fact 1.2.2. *Every closed submanifold of a Stein manifold is a Stein manifold.*

In fact there is the Embedding Theorem for Stein manifolds.

Fact 1.2.3. *Every Stein manifold can be holomorphically embedded as a closed submanifold into $\mathbb{C}^N$.*

Below we give one more equivalent definition of a Stein manifold in terms of plurisubharmonic function, that is often used in practice.

Definition 1.2.3. A function φ defined in an open set $\Omega \subset \mathbb{C}^n$ with values in $[-\infty, +\infty)$ is *plurisubharmonic* if

1. it is semicontinuous from above.
2. For an arbitrary z and $w \in \mathbb{C}^n$, the function $\tau \to \varphi(z + \tau w)$ is subharmonic in the part of $\mathbb{C}$ where it is defined.

Fact 1.2.4. *A function $\varphi \in C^2(\Omega)$ is plurisubharmonic if and only if*

$$\sum_{j,k=1}^{n} \partial^2 \varphi(z) / \partial z_j \partial \bar{z}_k w_j \bar{w}_k \geq 0, \tag{1.1}$$

where $z \in \Omega$, $w \in \mathbb{C}^n$.

Definition 1.2.4. Function φ is strictly plurisubharmonic if the inequality 1.1 is strict.

The notion of plurisubharmonicity does not depend on the choice of holomorphic coordinates. Therefore, it is well defined on all complex manifolds.

Fact 1.2.5. *A complex manifold X is a Stein manifold if and only if there exists a strictly plurisubharmonic function $\varphi \in C^{\infty}(X)$ such that*

$$\Omega_c = \{z|\ z \in X, \varphi(z) < c\} \Subset \Omega$$

for any real number c. The sets $\hat{\Omega}_c$ are the $\mathcal{O}(X)$-convex.

1.2.2 Complex foliations

Definitions 1.2.5-1.2.12 are from [IlYa]. They are scatted through out the text, so we provide them here for the convenience of the reader. Definition 1.2.13, 1.2.14 can be found in [V06],[Ch86] correspondingly.

Definition 1.2.5. Let $\mathcal{F}$ be a foliation of a complex manifold X. Let $\gamma : [0, 1] \to X$ be a path on X. Let T_0 and T_1 be two transversal sections to $\mathcal{F}$, passing through $\gamma(0)$ and $\gamma(1)$ respectively. Then for any initial point $x \in T_0$, close to $\gamma(0)$, leaf-wise curves, starting from x, and staying close to γ, and arriving to T_1, arrive at a well defined point $\Delta_\gamma(x)$. Thus, we obtain a map $\Delta_\gamma(x)$, which we call *holonomy map*.

Definition 1.2.6. If $\gamma : [0, s] \to X$ is a closed curve, and T is a transversal section to $\mathcal{F}$, passing through $\gamma(0)$. The map $\Delta_\gamma : T \to T$ is called the *holonomy map* as well.

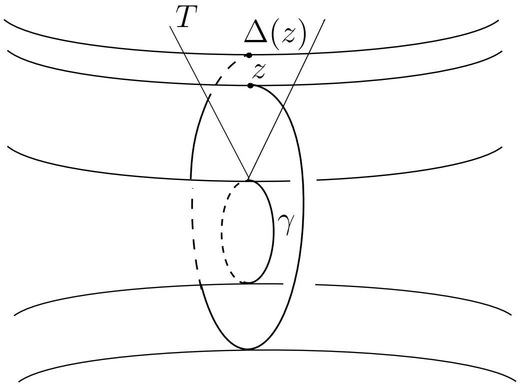


Figure 1.3: A holonomy map

The holonomy map is a holomorphic map. If one changes the transversal section, the holonomy map gets conjugated by a holomorphic map.

A complex vector field determines a one-dimensional foliation in the complement to its singular locus. In this paper we work only with one-dimensional foliations. Later on the word "foliation" means one-dimensional foliation.

Definition 1.2.7. *A singular holomorphic foliation* in complex analytic manifold is a holomorphic foliation $\mathcal{F}$ in the complement $U \backslash \Sigma$ to an analytic subset Σ, $\mathrm{codim} \Sigma \geq 2$, called the *singular locus* of $\mathcal{F}$.

Note that singular holomorphic foliations are locally generated by holomorphic vector fields. For the proof see [Il72], [IlYa]. There it is proved for domains in $\mathbb{C}^n$, but the proof works for any complex analytic manifold.

Definition 1.2.8. *A complex cycle* is a nontrivial free homotopy class of loops on a leaf of foliation.

Definition 1.2.9. A complex cycle γ is called *isolated* if it corresponds to an isolated fixed point of its holonomy map. It is called *identical* if the corresponding holonomy map is identical.

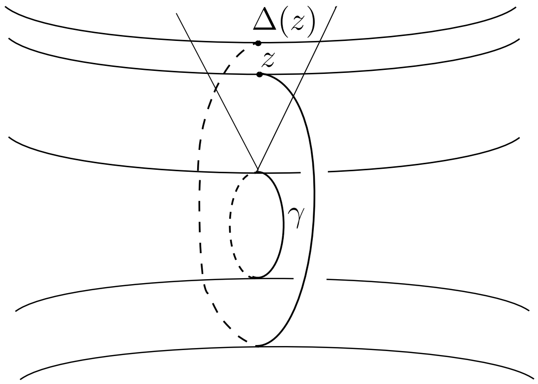


Figure 1.4: An isolated cycle

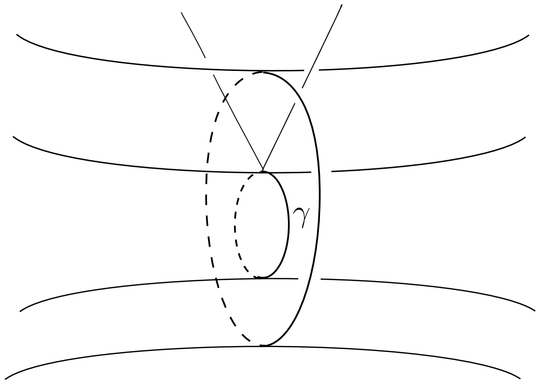


Figure 1.5: An identical cycle

For foliations of 2-dimensional Stein manifolds all cycles are either isolated or identical. For higher dimensions there are also non-isolated cycles, i.e. non-identical cycles that belong to holomorphic families.

Definition 1.2.10. A complex cycle is *hyperbolic* if its holonomy map is hyperbolic, i.e. the eigenvalues of its derivative do not belong to the unit circle.

Definition 1.2.11. A singular point is called *complex hyperbolic* if it is non-degenerate and the ratio of any two eigenvalues is not real.

In this thesis we work only with complex hyperbolic singular points. So we reserve the word "hyperbolic" to complex hyperbolicity.

Note that complex hyperbolicity plays similar role for the theory of complex vector fields as hyperbolicity for the theory of real vector fields. In particular, if the point is complex hyperbolic, then the phase portrait of the vector field in the neighborhood of a singularity is homeomorphic to the phase portrait of its linearization [Ch86],[Guc72]. See ([IlYa],section 29) for thorough consideration of properties of complex hyperbolic points.

Definition 1.2.12. A *complex separatrix* of a singular holomorphic foliation $\mathcal{F}$ at a singular point $a \in \Sigma(\mathcal{F})$ is a local leaf $L \subset (U,a)\backslash\Sigma$, whose closure $L \cup a$ is a germ of an analytic curve.

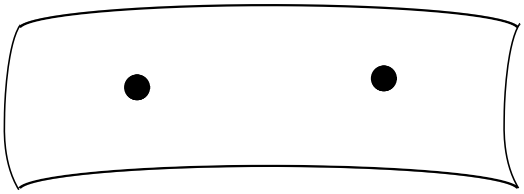


Figure 1.6: A heteroclinic saddle connection

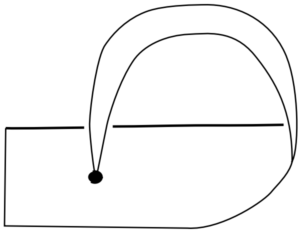


Figure 1.7: A homoclinic saddle connection

Definition 1.2.13. A *saddle connection* is common a separatrix of two singular points.

Definition 1.2.14. Suppose a is a hyperbolic singular point of the foliation $\mathcal{F}$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of a. Let l be a line passing through the origin in $\mathbb{C}$. Let $\lambda = (\lambda_{i_1}, \ldots, \lambda_{i_k})$ be the eigenvalues of a that lie on one side of the line l. Let α_λ be a subspace spanned by the eigenspaces of all elements of λ. The local *strongly invariant manifold* M_λ^{loc} is a manifold tangent to α_λ. The global strongly invariant manifold M_λ is obtained by taking the union of leaves that belong to the local strongly invariant manifold.

Strongly invariant manifolds exist [IlYa]. The proof can be easily modified to show that they depend holomorphically on a vector field (on a foliation).

Suppose that v is a vector field that determines the foliation locally. Strongly invariant manifolds are stable and unstable manifolds of the time-one map Φ_{cv}^1 of the vector field cv, where $c \in \mathbb{C}^*$ is taken so that l becomes the imaginary axis. If one considers the real flow of the vector field cv, then locally strongly invariant manifolds coincide with stable and unstable manifolds [Ch86].

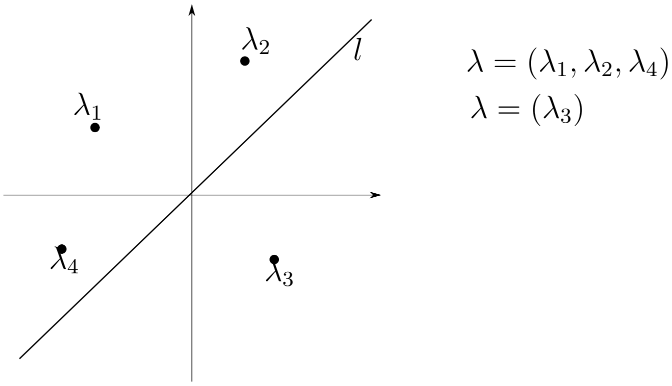


Figure 1.8: A line, separating eigenvalues.

1.2.3 Holomorphic vector bundle associated to a foliation

Take a 1-dimensional singular holomorphic foliation $\mathcal{F}$ of a Stein manifold M. One can naturally associate a linear bundle $B_{\mathcal{F}}$ to $\mathcal{F}$.

Notice that a foliation by holomorphic curves is locally determined by a holomorphic vector field [IlYa]. Consider a covering of a Stein manifold by open contractible sets U_i. On each set U_i the foliation is determined by a holomorphic vector field v_i. For a pair of intersecting sets U_i and U_j define a function $g_{ij} = v_i/v_j$. This function is well-defined on $(U_i \cap U_j) \setminus \{v_j = 0\}$. The set $\{v_j = 0\}$ has codimension 2. Therefore, g_{ij} can be extended to $U_i \cap U_j$.

The same way $g_{ji} = v_j/v_i$ can be extended to a well-defined function on $U_i \cap U_j$.

$$g_{ij}g_{ji} = 1 \quad \Rightarrow \quad g_{ij}\big|_{U_i \cap U_j} \neq 0$$

The set of functions $\{g_{ij}\}$ form a 2-cocycle, therefore, they define a linear bundle.

Lemma 1.2.1. 1-*dimensional singular holomorphic foliation of a Stein manifold X with $H^2(X, \mathbb{Z}) = 0$ is determined by a holomorphic vector field.*

Proof. A linear bundle is an element of $H^1(X, \mathcal{O}^*)$. Since X is a Stein manifold, $H^1(X, \mathcal{O}^*) = H^2(X, \mathbb{Z})$. $H^2(X, \mathbb{Z}) = 0$ by assumption. Thus, a bundle defined by g_{ij}, is trivializable and the foliation can be determined by a vector field globally. $\square$

In particular, the foliations of $\mathbb{C}^n$ are globally determined by vector fields.

Lemma 1.2.2. *1-dimensional singular holomorphic foliation $\mathcal{F}$ of a Stein manifold X is determined by a global section of the vector bundle $TX \otimes B_{\mathcal{F}}$.*

Proof. This is follows from the construction of $B_{\mathcal{F}}$. □

1.2.4 Topology of the uniform convergence on compact nonsingular sets

First, we provide the description of the topology for C^n. In this description we follow [GKK]. As we pointed out in the previous subsection, the foliation of $\mathbb{C}^n$ is globally determined by a vector field. But we do not need the n-tuple $(f_1, \dots, f_n)$ to define the foliation. We only need the direction of the complex vector, tangent to the leaves. The natural and the standard way to define the topology on the space of foliations is the following. A basis of neighborhoods of the foliation $\mathcal{F}$ is formed by

$$U_{R,\varepsilon,\delta} = \left\{ \begin{matrix} \mathcal{G} |\ \mathcal{G} \text{ is nonsingular in } K_{\varepsilon,R} = B_R(0) \backslash U_\varepsilon(\Sigma(\mathcal{F})) \text{ and the tangent direc-} \\ \text{tions to the foliations } \mathcal{F} \text{ and } \mathcal{G} \text{ are } \varepsilon\text{-close on } K_{\varepsilon,R}, \end{matrix} \right\} \tag{1.2}$$

where $\varepsilon, \delta > 0$, $B_R(0)$ is the ball of radius R with the center at the origin, $\Sigma(\mathcal{F})$ is the singular locus of $\mathcal{F}$.

So, informally speaking, two foliations are close if their tangent directions are uniformly close on a ball of a large radius except for some small neighborhood of the singular locus.

The map from the space of vector fields, with the topology of uniform convergence on compact sets, to the space of foliations, is continuous.

The integral representation of derivatives of a holomorphic function implies:

If two functions are close on a disk, then they are close with their derivatives up to the order r on a smaller disk.

Therefore, if in holomorphic case one requires that $\mathcal{F}$ and $\mathcal{G}$ in 1.2 are ε-close together with their derivatives up to the order r, one gets the same topology.

We fix a compact exhaustion:

$$K_1 \Subset \cdots \Subset K_n \cdots \Subset X,$$

where $K_1, \ldots, K_n$ are compact subsets of X, closures of open connected subsets of X; $\cup_n K_n = X$.

We fix metrics d_1 on X and d_2 on the projectivization of its tangent bundle PTX.

A basis of neighborhoods of the foliation $\mathcal{F}$ is formed by

$$U_{n,\varepsilon,\delta} = \begin{Bmatrix} \mathcal{G}|\ \mathcal{G} \text{ is nonsingular in } K_{\varepsilon,n} = K_n \backslash U_\varepsilon(\Sigma(\mathcal{F})) \text{ and the tangent directions} \\ \text{to the foliations } \mathcal{F} \text{ and } \mathcal{G} \text{ are } \varepsilon\text{-close on } K_{\varepsilon,n}, \end{Bmatrix}$$

Note that the obtained topology does not depend on the choice of the compact exhaustion and the choice of metrics d_1 and d_2.

Note that the set of foliations of X has countably many connected components, parametrized by Chern classes of the linear bundles, associated to the foliations.

The set of sections of $TX \otimes B_{\mathcal{F}}$ is equipped with the topology of uniform convergence on compact sets. The map from the space of sections to the space of foliations is continuous.

1.3 History of the subject.

1.3.1 Polynomial foliations.

Polynomial foliations of $\mathbb{C}^2$ have been extensively studied since the times of Poincaré. The detailed survey of the theory of polynomial foliations is provided in [Il02],[Il04]. The connection with Hilbert's 16th problem is one of the motivations to study the subject.

The polynomial differential equation in $\mathbb{C}^2$ is an equation of the form:

$$\frac{dx}{dy} = \frac{P_n(x,y)}{Q_n(x,y)},$$

where P_n, Q_n are polynomials of degree n.

The polynomial foliation is determined by a pair (P_n, Q_n), defined up to a multiplication by a non-zero function.

There has been significant progress in understanding the global topology of the phase portrait of a foliation.

The density was first proved by Khudai-Verenov [Kh62], rigidity and countable number of limit cycles was established by Ilyashenko [Il78]. Many others have contributed to this subject, proving these results with stronger notions of genericity.

A leaf of a foliation is a Riemann surface and in general it can have fundamental group with any number of generators. The following question was asked by Anosov in 60's and still remains unsolved.

Problem 1.3.1. *What is the topological type of a leaf for a generic polynomial foliation?*

We expect the answer to be the same as in the holomorphic case. Though the topology of the leaves of a generic foliation is not known, the conformal type of the leaves for a generic foliation was described by Candel-Gomez-Mont [CGM95]. The result was later improved by Lins Neto [LN94], and Glutsyuk[G94].

Theorem 1.3.1. *[G94], [LN94] Any leaf of a generic polynomial foliation of degree n is hyperbolic.*

1.3.2 Analytic foliations

In my paper [F06] I give the answer to the Anosov's question in the case of holomorphic foliations of $\mathbb{C}^n$. I proved the following theorem:

Theorem 1.3.2. *For a generic holomorphic foliation of* $\mathbb{C}^2$ *all leaves except for not more than a countable number are topological disks, the rest are topological cylinders. Moreover, a generic foliation is complex Kupka-Smale.*

Golenishcheva-Kutuzova [GK06] showed that for a generic foliation countable many cylinders do exist.

Problem 1.3.2. *Prove that for a generic singular* 1*-dimensional holomorphic foliation of a Stein manifold countable number of cylinders exist.*

Problem 1.3.3. *What is the conformal type of leaves of a generic singular* 1*-dimensional holomorphic foliation of a Stein manifold?*

We expect that the technique from [CGM95], [LN94], [G94] can be adjusted to attack the problem. See the paper [Il08] for a vast discussion of open problems.

The Kupka-Smale property for a generic polynomial automorphism of C^n was established by Buzzard in [B98].

1.4 Local elimination of degenerate objects.

As we pointed out in the introduction one can not eliminate a homoclinic saddle connection locally in a flow-box. Rather than that one needs to perturb the foliation in the neighborhood of the separatrix loop. This leads us to considering degenerate objects.

We consider the following degenerate objects:

1. A non-hyperbolic loop γ on the leaf of foliation $\mathcal{F}$;
2. A pair of loops γ_1, γ_2 that belong to the same leaf L; are not null homologous on L; and are not multiples of the same cycle in the homology group of L.

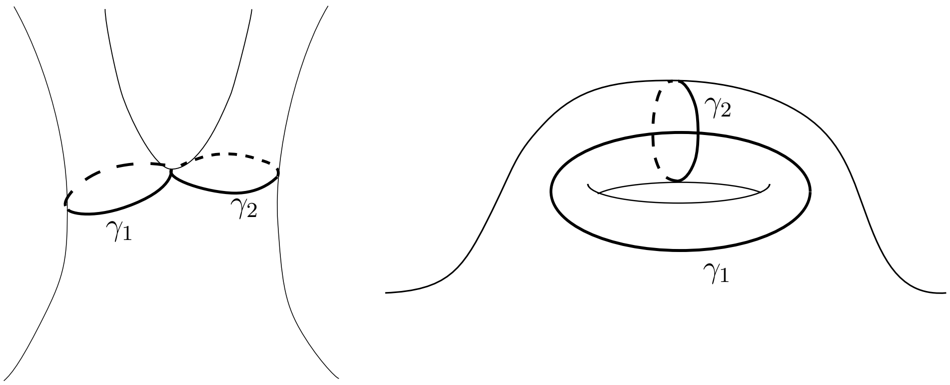


Figure 1.9: A pair of cycles

3. A path γ on a saddle connection, that connects two different singular points a_1 and a_2.

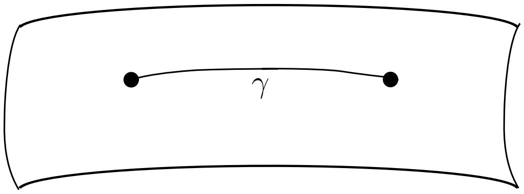


Figure 1.10: A path on a saddle connection

4. A loop γ on a homoclinic saddle connection, that connects a singular point a with itself.

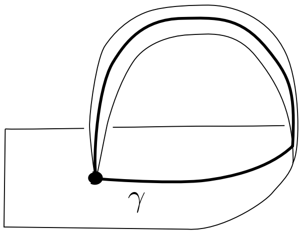


Figure 1.11: A loop on a homoclinic saddle connection

5. A loop γ on a separatrix that is not homotopic to a multiple of the loop around the singular point a.

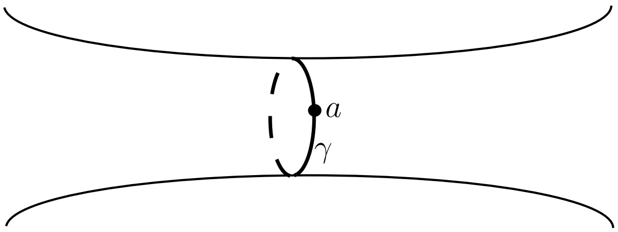


Figure 1.12: A loop on a separatrix.

6. A pair of paths γ_1 and γ_2, so that γ_1 and γ_2 belong to the strongly invariant manifolds M_1 and M_2 of singular points a_1 and a_2 correspondingly; and connect a_1 and a_2 correspondingly with a point of non-transversal intersection of M_1 and M_2.

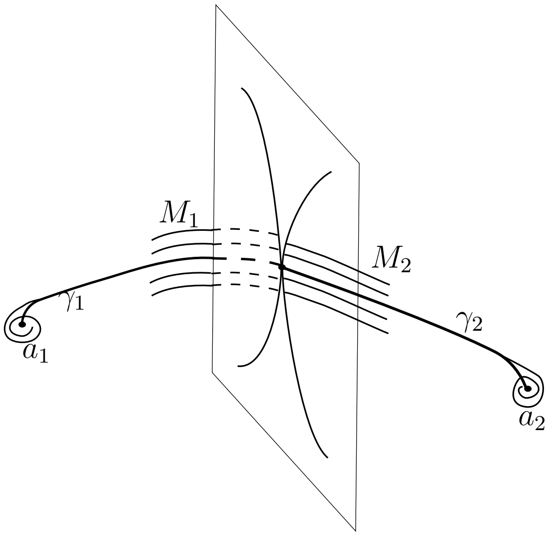


Figure 1.13: A non-transversal intersection of strongly invariant manifolds.

7. A loop $\gamma = \gamma_1 \cup \gamma_2$, so that γ_1 and γ_2 belong to strongly invariant manifolds M_1 and M_2 correspondingly of a singular point a; and connect a with a point of non-transversal intersection of M_1 and M_2 correspondingly. (We assume that in a neighborhood of the point a, $M_1^{loc} \cap M_2^{loc} = a$.)

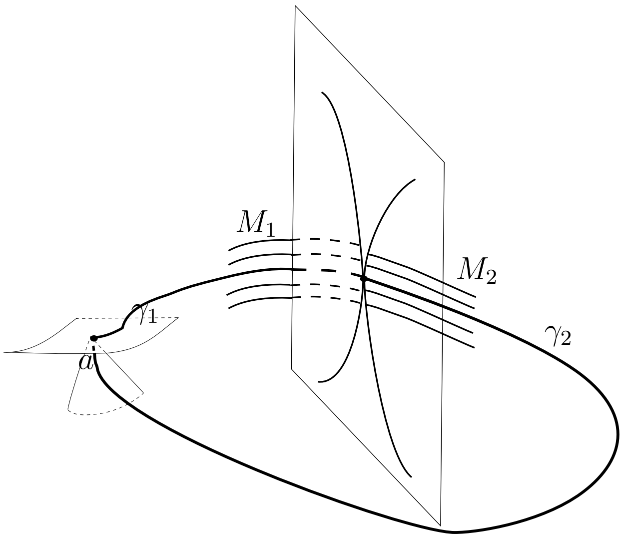


Figure 1.14: A homoclinic non-transversal intersection of st. inv. manifolds.

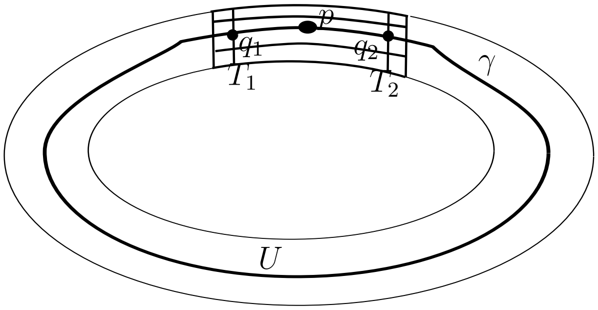
In this section we find a neighborhood of a degenerate object and and a family of holomorphic foliations in this neighborhood that eliminate the degenerate object in the neighborhood.

Our technique allows us to do that only if the degenerate object is holomorphically convex. We expect, though, it should be possible to carry out for any smooth enough curve. In Section 1.6 we show that all the degenerate objects, satisfying this additional condition, can be eliminated simultaneously. Moreover, if a foliation does not have holomorphically convex degenerate objects, then it does not have degenerate objects. Note that in all cases degenerate objects are paths or closed curves. In the following two subsections we describe a general procedure of constructing a local family of foliations in a neighborhood of a holomorphically convex curve.

Let U be a neighborhood of the curve. First, we allow not only the foliation, but the neighborhood itself to change with the parameter λ. We get a family of foliations $\mathcal{F}_\lambda$ on manifolds U_λ. Then we find the way to 'project' U_λ to some neighborhood of the degenerate object. Thus, we produce a family of foliations in the neighborhood of the degenerate object that breaks it.

The following lemma summarizes the results of the following two subsections.

Let γ be a holomorphically convex curve on a Stein manifold X, endowed with a foliation $\mathcal{F}_0$. Fix a point $p \in \gamma$, assume that $p \notin \Sigma(\mathcal{F})$. Assume that in a neighborhood of p the curve γ belongs to a leaf L of $\mathcal{F}_0$. Take a flow-box in a neighborhood of the point p. Fix coordinates $(z_1, \ldots, z_{n-1}, t)$ in a neighborhood of p, such that t is a coordinate along the foliation, $z(p) = 0$. Let $\alpha \subset \gamma$ be a small arc, a neighborhood of p on γ. Take a pair of points $q_1, q_2 \in \gamma \backslash \alpha$, that lie on different sides of α and belong to the flow-box. Let T_1, T_2 be transversal sections to $\mathcal{F}_0$ that pass through q_1, q_2. Functions $(z_1, \ldots, z_{n-1})$ work as coordinates on T_1, T_2.



,

Figure 1.15:

Lemma 1.4.1. *Fix a holomorphic on λ family of germs of biholomorphisms*

$$\Phi_\lambda : \left(\mathbb{C}^{n-1}, 0\right) \to \left(\mathbb{C}^{n-1}, 0\right), \quad \Phi_0 = Id.$$

Then there exists a neighborhood U of γ with a retraction $\rho : U \to \gamma$ and a family of foliations $\mathcal{F}_\lambda$ on U that depends holomorphically on λ satisfying the following conditions:

1. *outside $\rho^{-1}(\alpha)$ $\mathcal{F}_\lambda$ is biholomorphic to $\mathcal{F}_0$. More precisely, there exists a holomorphic on λ family of maps $\pi_\lambda : \left(U \backslash \rho^{-1}(\alpha)\right) \to X$, which are biholomorphisms to their images, such that π_λ maps the leaves of $\mathcal{F}_0$ to the leaves of $\mathcal{F}_\lambda$, $\pi_0 = Id$;*

2. *The holonomy map inside the flow-box along the foliation $\mathcal{F}_\lambda$ between T_1 and T_2 is biholomorphically conjugate to Φ_λ, more precisely, in coordinates $(z_1, \ldots, z_{n-1})$ on*

T_1, T_2 it is $(\pi^z_\lambda)^{-1} \circ \Phi_\lambda \circ \pi^z_\lambda$, where π^z_λ and $(\pi^z_\lambda)^{-1}$ are first $(n-1)$ coordinates of π_λ and π^{-1}_λ correspondingly.

This lemma is supposed to mimic the smooth case, when we change the foliation only in the flow-box. In the holomorphic case this is not possible. Therefore, we need to adjust everything by the map π_λ.

1.4.1 Regluing.

Let γ be a path or a loop in a Stein manifold X. Let us fix a $p \in \gamma$.

Let $\Phi_\lambda : \Omega \to \mathbb{C}^{n-1}$ be a family of biholomorphisms that depend holomorphically on 1-dimensional parameter λ, $\Omega \subset \mathbb{C}^{n-1}$, $0 \in \Omega$, $\Phi_0 = Id$.

We start by constructing manifolds U_λ. They are obtained by regluing U in a flow-box around a point p. First, we describe the procedure informally and point out the technical difficulties that arise. Then we repeat the description paying attention to the technical difficulties.

We take a neighborhood U that can be retracted to γ. Let $\hat{U}$ be a complex manifold that projects one-to-one to U everywhere except for a flow box around a point p and projects two-to-one to this flow box. U is obtained from $\hat{U}$ by gluing the points in the flow-boxes using identity map. U_λ is obtained from $\hat{U}$ by gluing the points in the flow-boxes by using the map Φ_λ. The problem is that Φ_λ is not an isomorphism from the flow-box to itself. Thus, extra caution is needed to make U_λ Hausdorff. In the rest of the section we describe these precautions.

First, we choose a bigger neighborhood W that can be retracted to γ. Let ρ denote the retraction. Fix a point $p \in \gamma$ and an arc $\alpha \subset \gamma$, $p \in \alpha$. Take a transversal section T to the foliation $\mathcal{F}_0$ at a point p. Let $(z_1, \dots, z_{n-1})$ be coordinates on T. Let t be a coordinate along the leaves of the foliation $\mathcal{F}$.

Let $\hat{W}$ be a connected complex manifold that projects one-to-one on $U \backslash \rho^{-1}(\alpha)$ and two-to-one on $\rho^{-1}(\alpha)$. Let π_1 be the first projection and π_2 the second.

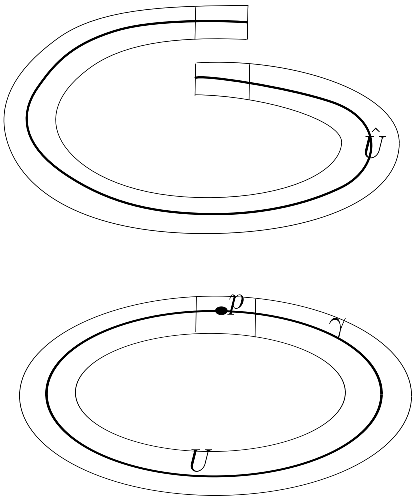


Figure 1.16: Regluing

Let V be a flow-box around a point p in W. We assume $V \subset \rho^{-1}(\alpha)$. Fix coordinates $(z_1, \ldots, z_{n-1}, t)$ in V. We take V small enough so that $(\Phi_\lambda, \mathrm{Id})$ is a well-defined map on V and is a biholomorphism to its image. Let $V_1 = \pi_1^{-1}(V)$, $V_2 = \pi_2^{-1}(V)$.

Let $T_c \subset W$ be a a tube of points that are at distance c from the preimage of $\gamma \backslash \alpha$. Let $\hat{T}_c \subset \hat{W}$ be a tube of points that are at the distance c from the preimage of $\gamma \backslash \alpha$. Take c small enough.

Take $U = T_c \cup V$, $\hat{U} = V_1 \cup V_2 \cup \hat{T}_c$. Note that U can obtained from $\hat{U}$ by gluing the points from V_1 and V_2 that project to the same point in W.

Let $V_2^\lambda = \pi_2^{-1}((\Phi_\lambda, Id)(V))$

Let $\hat{U_\lambda} = V_1 \cup \hat{T}_c \cup V_2^\lambda$. U_λ is a space obtained from $\hat{U}_\lambda$ by gluing V_1 and V_2^λ by the map $(\Phi_\lambda, \mathrm{Id})$. The space U_λ inherits complex structure. The following standard lemma gives the necessary condition for the obtained space to be Hausdorff.

Lemma 1.4.2. *Let X_1 be a an open subspace of X_2, $X_1 \subset X_2 \subset X$, where X is a Hausdorff metric space. Let $\varphi : X_1 \to \varphi(X_1) \subset X_2$ be a homeomorphism to its image, that, moreover, extends to be homeomorphism to the boundary. Let $\varphi(\partial X_1 \cap X_2) \subset \partial\varphi(X_1) \backslash X_2$, $\varphi^{-1}(\partial\varphi(X_1) \cap X_2) \subset \partial X_1 \backslash X_2$ and $\overline{\varphi(X_1)} \cap \overline{X_1} = \emptyset$. Then the space $X_2/\sim$, where $x \sim \varphi(x)$, $x \in X_1$, is a Hausdorff space.*

One can take c and λ small enough so that triple $X_1 = V, X_2 = \hat{U}_\lambda, \varphi = \Phi_\lambda$ satisfies the hypothesis of the lemma. Therefore, U_λ is a complex manifold.

$$\hat{\mathcal{U}} = \{(u, \lambda) \in \hat{W} \times \Lambda |\ u \in V_1 \cup T_c \cup V_2^\lambda, \lambda \in \Lambda\}$$

$$\mathcal{U} = \hat{\mathcal{U}} / \sim, (u, \lambda) \sim ((\Phi_\lambda, \mathrm{Id})(u), \lambda)\,, \text{ where } u \in V_1, \lambda \in \Lambda$$

Note that regluing is a local procedure, we do it in a neighborhood of a point.

1.4.2 Projection. Siu's Theorem.

The goal of this step is to prove that one can take a small neighborhood of γ in U_λ and project it biholomorphically to a neighborhood of γ in U.

Suppose γ is holomorphically convex. One can choose a neighborhood U_1 of γ, $U_1 \subset U$, such that U_1 is an analytic polyhedron, therefore, a Stein manifold [GR65].

To produce a neighborhood of U_1 in $\mathcal{U}$ we use Siu's theorem [Siu76]:

Theorem 1.4.1. *Suppose X is a complex space and A is a subvariety of X. If A is Stein, then there exists an open neighborhood ω of A in X such that ω is Stein.*

Thus, there is a neighborhood of U_1 in $\mathcal{U}$ that is Stein. It can be embedded to $\mathbb{C}^N$ for some N. We want to find a biholomorphism from a neighborhood of γ in U_λ to some neighborhood of γ in U. The next lemma states that one can take a projection along one of the linear subspaces of $\mathbb{C}^N$ of dimension $(N - n)$ to be such a biholomorphism.

Lemma 1.4.3. *There exists a linear $(N-n)$-subspace $\alpha \subset \mathbb{C}^N$ such that all affine subspaces $\alpha_x \subset \mathbb{C}^N$ parallel to α passing through point $x \in \gamma$ are:*

a) transverse to U;

b) pass through only one point on γ.

Proof. The set of all $(N-n)$-subspaces of $\mathbb{C}^N$ is $n(N-n)$-dimensional complex manifold $Gr(N-n,N)$.

Elements of $Gr(N-n,N)$ that are not transverse to a given subspace of complementary dimension form a codimension 1 complex (singular) subvariety. Path γ is 1-dimensional real manifold. Therefore, subspaces that do not satisfy (a) form a subvariety of $Gr(N-n,N)$ of real codimension 1.

A couple of points on γ form a real 2-dimensional manifold. Linear subspace parallel to those that pass through two given points in $\mathbb{C}^N$ form $(n(N-n-1))$-dimensional manifold. Therefore, subspaces that do not satisfy (b) form a submanifold of $Gr(N-n,N)$ of real codimension $2(n-1)$.

Since $n \geq 2$, a $(N-n)$-subspace α, that satisfies conditions (a) and (b), exists.

□

Consider the projection π_λ along α from U to U_λ. By lemma 1.4.3 this map is well-defined and biholomorphism to its image in some neighborhood $U_2 \subset U$ of γ for all small λ. Note that away from $\rho^{-1}(\alpha)$ we can identify U and U_λ. Having this identification in mind, $\pi(\lambda)$ is a desired map.

1.4.3 Removal of a non-hyperbolic cycle.

In this subsection we apply the technique developed in the previous two subsections to break a non-hyperbolic cycle, in assumption that the cycle is holomorphically convex.

Lemma 1.4.4. *Let γ be a representative of a non-hyperbolic cycle of the foliation $\mathcal{F}_0$, that is holomorphically convex. Then there exist a neighborhood U of γ and a holomorphic family $\mathcal{F}_\lambda$ of foliations in this neighborhood, so that for $\lambda \in V \backslash R$ all the cycles on the leaves of foliation $\mathcal{F}_\lambda$, close to γ, are hyperbolic. V is a neighborhood of $0 \in \mathbb{C}$ and R is a real-analytic 1-dimensional subset in V, $0 \in R$.*

Proof. Take a point $p \in \gamma$ and a transversal section T to $\mathcal{F}$, $p \in T$. Let $\Delta_\gamma : (T,p) \rightarrow$

(T, p) be the corresponding holonomy map. The cycle γ is hyperbolic by the definition if and only if all the eigenvalues of Δ_γ lie not on the unit circle.

First, we provide a specific perturbation of Δ_γ that has hyperbolic fixed points only.

The following lemma is the standard fact:

Lemma 1.4.5. *There exists a diagonal* $n \times n$ *matrix* D *and* $a \in \mathbb{C}^n$ *such that the map* $\Delta_\gamma(z) + \lambda(Dz + a)$ *is well-defined and has hyperbolic fixed points only for all* $\lambda \in V \backslash R$, *where* V *is a neighborhood of* 0, R *is a real-analytic curve,* $0 \in R$.

Take a, D such that Lemma 1.4.5 is satisfied.

Apply Lemma 1.4.1 to the cycle γ, the point p and the family of biholomorphisms $\Phi_\lambda = Id + \lambda(Dz + a)$. The map $\Delta_\gamma^\lambda = \pi_\lambda^{-1} \circ (\Delta_\gamma + \lambda(\Delta z + a)) \circ \pi_\lambda$ is the holonomy map along γ for the foliation $\mathcal{F}_\lambda$. For all λ outside a one-dimensional real-analytic curve R the map Δ_γ^λ has hyperbolic fixed points only on T. □

1.4.4 Splitting cycles to different leaves

Lemma 1.4.6. *Hyperbolic cycles persist under the perturbation.*

Proof. Let γ be a hyperbolic cycle on the leaf of foliation $\mathcal{F}_0$. Let T be a transversal section at a point $p \in \gamma$. Let $\mathcal{F}_\lambda$ be a holomorphic family of foliations. Let Δ_γ^λ be a holonomy map along γ for the foliation $\mathcal{F}_\lambda$ defined on a transversal section T. By the inverse function theorem Δ_γ^λ has one isolated fixed point in a neighborhood of a point p on T, and this isolated fixed point is hyperbolic. □

Let γ^λ be the perturbation of γ.

Lemma 1.4.7. *Let* γ_1, γ_2 *be representatives of hyperbolic cycles, lying on the same leaf* L *of a foliation* $\mathcal{F}_0$. *One can assume that they pass through the same point* p. *Assume that the union* $\gamma = \gamma_1 \cup \gamma_2$ *is holomorphically convex. Then there exists a neighborhood* U *of* γ *and a holomorphic family of foliations* $\mathcal{F}_\lambda$ *in this neighborhood, so that for* $\lambda \neq 0$, γ_1^λ *and* γ_2^λ *lie on different leaves of foliation* $\mathcal{F}_\lambda$.

Proof. Let $T \ni p$ be a transversal section to a leaf L. Let $\Delta_{\gamma_1}, \Delta_{\gamma_2} : T \to T$ be the holonomy maps along γ_1 and γ_2 correspondingly. The fact that γ_1 and γ_2 lie on the same leaf is equivalent to Δ_{γ_1} and Δ_{γ_2} having a common zero at a point p.

Let $q \in \gamma_1 \backslash \gamma_2$, and is not a point of self-intersection. Apply Lemma 1.4.1 to the curve γ, the point q, and the family of biholomorphisms $\Phi_\lambda = z + \lambda$. Then $\pi_\lambda^{-1} \circ \Delta_{\gamma_2} \circ \pi_\lambda$ is a holonomy map along γ_2. Let T_1 be a transversal section to the foliation $\mathcal{F}_0$ in a point q. The holonomy map along γ_1 for the foliation $\mathcal{F}_0$ can be written as a composition $\Delta_{\gamma_1} = \Delta_2 \circ \Delta_1$, where Δ_1 is a holonomy map from transversal section T to T_1, Δ_2 is a holonomy map from T_1 to T. Then the holonomy map along γ_1 for the foliation $\mathcal{F}_\lambda$ is $\pi_\lambda^{-1} \circ \Delta_2 \circ \Phi_\lambda \circ \Delta_1 \circ \pi_\lambda$.

$\pi_\lambda^{-1}(p)$ is an isolated fixed point for the holonomy map along γ_2 and is not a fixed point for the holonomy map along γ_1. Thus, cycles split to different leaves.

□

1.4.5 Elimination of a saddle connection.

Lemma 1.4.8. *Let γ be a path on a saddle connection S of the foliation $\mathcal{F}_0$, connecting different singular points a_1 and a_2. Assume that γ is holomorphically convex. Then there exists a neighborhood U and a holomorphic family of foliations $\mathcal{F}_\lambda$ in this neighborhood, so that for $\lambda \neq 0$ there does not exist a saddle connection S_λ of $\mathcal{F}_\lambda$ with a path on it, close to γ.*

Proof. Take a point $p \in \gamma$, $p \neq a_1$, $p \neq a_2$. Take a transversal section T at p to the foliation $\mathcal{F}_0$. Also take transversal sections T_1, T_2 to $\mathcal{F}_0$ at points $q_1 \in \gamma$, $q_2 \in \gamma$, close to singular points a_1 and a_2 correspondingly. Fix coordinates $(z_1, \dots, z_{n-1})$ on transversal sections T, T_1, T_2 so that the holonomy map along γ is identity. Let Δ_1 be the holonomy map from T_1 to T. Let Δ_2 be the holonomy map from T_2 to T.

Apply Lemma 1.4.1 to the path γ, the point p, taking $\Phi_\lambda = z + \lambda$. Let $\Delta_1^\lambda : T_1 \to T$, $\Delta_2^\lambda : T_2 \to T$ be holonomy maps for the foliation $\mathcal{F}_\lambda$. The saddle connection persists if the holonomy maps Δ_1, Δ_2 have a common fixed point.

$$\Delta_1^\lambda = (\pi_\lambda^z)^{-1} \circ \Delta_1 \circ \pi_\lambda^z;$$

$$\Delta_2^\lambda = (\pi_\lambda^z)^{-1} \circ (\Delta_2 + \lambda) \circ \pi_\lambda^z$$

Therefore, for $\lambda \neq 0$, $\Delta_1^\lambda(\pi_\lambda^z(p)) = \pi_\lambda^z(p)$, $\Delta_2^\lambda(\pi_\lambda^z(p)) \neq \pi_\lambda^z(p)$. Thus, they have different fixed points in a neighborhood of the point p. ☐

Lemma 1.4.9. *Let γ be a loop on a homoclinic saddle connection S of the foliation $\mathcal{F}$, going from a singular point a to itself. Assume that one end of γ belongs to one local component of γ, the other end belongs to the other connected component. Suppose that γ is holomorphically convex. Then there exist a neighborhood U and a family of holomorphic foliations $\mathcal{F}_\lambda$ in this neighborhood, so that for $\lambda \neq 0$ there does not exist a homoclinic saddle connection S_λ of $\mathcal{F}_\lambda$ with a path on it close to γ.*

Proof. The proof is the same as in the Lemma 1.4.8 ☐

1.4.6 Elimination of a cycle on a separatrix.

Let a be a hyperbolic singular point, S its separatrix.

Lemma 1.4.10. *Let γ be a cycle on $S \cup a$, that pass through a point a. Assume that γ is holomorphically convex, then there exists a neighborhood U of γ and a family of foliations $\mathcal{F}_\lambda$ in this neighborhood, such that for $\lambda \neq 0$, there does not exist a separatix S_λ of the foliation $\mathcal{F}_\lambda$ with a cycle close to γ on it.*

Proof. Take a point $p \in \gamma$, $p \neq a$. Apply Lemma 1.4.1 to the point p, the cycle γ, taking $\Phi_\lambda(z) = z + \lambda$. On S_λ there is no cycle close to γ. ☐

1.4.7 Elimination of a non-transversal intersection of strongly invariant manifolds.

Critical points and strongly invariant manifolds depend holomorphically with respect to parameter [IlYa].

Lemma 1.4.11. *Let M_1, M_2 be strongly invariant manifolds of two different hyperbolic singular points a_1, a_2 of a foliation $\mathcal{F}_0$. Let γ_1, γ_2 be paths on M_1, M_2 correspondingly that connect a_1, a_2 to a point p of a non-transversal intersection of M_1 and M_2. Suppose $\gamma = \gamma_1 \cup \gamma_2$ is holomorphically convex. Suppose there exists a leaf L of the foliation $\mathcal{F}_0$ so, that $\gamma \setminus \left(M_1^{loc} \cup M_2^{loc}\right) \subset L$. Then there exists a neighborhood U of γ and a holomorphic family of foliations $\mathcal{F}_\lambda$ such that for $\lambda \neq 0$ the strongly invariant manifolds M_1^λ, M_2^λ obtained by the continuation of local invariant manifold along γ intersect transversally in a neighborhood of the point p.*

Proof. Let T be a transversal section to the foliation in the point p, such that in a neighborhood of a point p strongly invariant manifolds M_1 and M_2 are biholomorphically equivalent to $m_1 \times D$, $m_2 \times D$, where $m_1 = M_1 \cap D_1$, $m_2 = M_2 \cap D_1$, D is a neighborhood of p on the leaf; D_1 is a neighborhood of p on the transversal section T. Fix coordinates $(z_1, \ldots, z_{n-1})$ on T. Apply Lemma 1.4.1 to the curve γ, the point p and $\Phi_\lambda = z + \lambda a$. Assume that points $q_1 \in \gamma_1$, $q_2 \in \gamma_2$.

Let M_1^λ, M_2^λ be the strongly invariant manifolds for the foliation $\mathcal{F}_\lambda$. Outside of the flow-box $M_1^\lambda = \pi_\lambda(M_1)$, $M_2^\lambda = \pi_\lambda(M_2)$.

In a neighborhood of the point p

$$T \cap M_2^\lambda = \pi_\lambda^z(m_2)$$

$$T \cap M_1^\lambda = \Phi_\lambda \circ \pi_\lambda^z(m_1).$$

Therefore, by Sard's Theorem, for almost all a they intersect transversally.

□

Lemma 1.4.12. *Let M_1, M_2 be strongly invariant manifolds of the hyperbolic singular point a of the foliation $\mathcal{F}$. Assume that $M_1^{loc} \cap M_2^{loc} = a$. Let $\gamma_1 \subset M_1$, $\gamma_2 \subset M-2$ be paths that connect a_1, a_2 with a point p of non-transversal intersection of M_1 and M_2. Suppose $\gamma = \gamma_1 \cup \gamma_2$ is holomorphically convex. Suppose that there exists a leaf L of foliation $\mathcal{F}_0$, so that $\gamma \backslash \left(M_1^{loc} \cup M_2^{loc}\right) \subset L$. Then there exists a neighborhood U of γ and a holomorphic family of foliations $\mathcal{F}_\lambda$ such that for $\lambda \neq 0$, M_1^λ, M_2^λ, obtained by continuing local strongly invariant manifolds along γ, intersect transversally in a neighborhood of the point p.*

Proof. The same arguments as in the previous lemma. □

1.5 From the local elimination of degenerate objects to the global elimination.

In this section we give the geometric conditions for degenerate objects to be holomorphically convex and show how to pass from a local foliation to a global one.

1.5.1 Approximation Theory.

Working in the category of smooth vector fields one can eliminate a non-transversality by perturbing the vector field only in a neighborhood of a non-transversality. In the holomorphic category there are no local perturbations allowed. However, approximation theory gives a way to work locally. In some cases you can perturb the local picture and then approximate your perturbation by a global one. In particular, for a holomorphic vector bundle on a Stein manifold holomorphic sections over a neighborhood of a holomorphically convex set can be approximated by global holomorphic sections. This follows from two theorems formulated below.

Theorem 1.5.1. ([H90], 5.6.2) *Let X be a Stein manifold and φ a strictly plurisubharmonic function in X such that $K_c = \{z\colon z \in X, \varphi(z) \leq c\} \Subset X$ for every real number c. Let B be an analytic vector bundle over X. Every analytic section of B over a neighborhood of K_c can then be uniformly approximated on K_c by global analytic sections of B.*

Theorem 1.5.2. ([H90], 5.1.6) *Let X be a Stein manifold, K a compact subset of X and U is an open neighborhood of holomorphic hull of K. Then there exists a function $\varphi \in C^\infty(X)$ such that*

1. *φ is strictly plurisubharmonic,*
2. *$\varphi < 0$ in K but $\varphi > 0$ in $X \backslash U$,*
3. *$\{z\colon z \in X, \varphi(z) < c\} \Subset X$ for every $c \in \mathbb{R}$.*

We need to approximate sections in a neighborhood of a loop or a path (i.e. a complex cycle, a path on a saddle connection). In the next section we state Stolzenberg's Theorem that describes the holomorphic hull of a curve and, more generally, of a collection of curves.

Lemma 1.5.1. *Let γ be a holomorphically convex degenerate object for the foliation $\mathcal{F}_\lambda$. Suppose that there exists a family $\mathcal{F}_\lambda$ in a neighborhood U of γ such that the degenerate object is eliminated for all $\lambda \in V \backslash R$, where V is a neighborhood of 0, $R \subset V$ is 1-dimensional real-analytic subset. Then there exists a family of global holomorphic foliations that eliminate the degenerate object.*

Proof. Let s_λ be a local section that determines $\mathcal{F}_\lambda$. One can assume that $1 \in V \backslash R$. By Theorems 1.5.1 and 1.5.2 there exists a global section S_1 that is ε-close to s_1 on U', where $\gamma \Subset U' \Subset U$. Therefore, family of foliations determined by $S_\lambda = S_0 + \lambda(S_1 - S_0)$ eliminate the degenerate object. □

1.5.2 Holomorphic convexity of a curve.

We recalled the definition of a holomorphic hull and gave examples of holomorphic hulls of curves in Subsection 1.2.1.

Consider a collection of C^1 - smooth real curves $\gamma_1, \dots, \gamma_m$ in $\mathbb{C}^N$. Their holomorphic hull is described by Stolzenberg's Theorem [St66]:

Theorem 1.5.3. *Let $\gamma = \gamma_1 \cup \dots \cup \gamma_m$. Then $h(\gamma) \backslash \gamma$ is a (possibly empty) one-dimensional analytic subset of $\mathbb{C}^N \backslash \gamma$.*

An easy corollary from this theorem is that the result is true if one replaces $\mathbb{C}^N$ by a Stein manifold X.

Let X be a Stein manifold, $\gamma_1, \dots, \gamma_m \subset X$ be real C^1 - smooth curves. Let $\gamma = \gamma_1 \cup \dots \cup \gamma_m$. Let $h_X(\gamma)$ be the holomorphic hull of γ in X. Then the following lemma holds.

Corollary 1.5.1. *$h_X(\gamma) \backslash \gamma$ is a (possibly empty) one-dimensional analytic subset of $X \backslash \gamma$.*

Proof. There exists a proper embedding of the Stein manifold X to $\mathbb{C}^N$ for some large enough N ([H90], theorem 5.3.9). Let $h(\gamma)$ be the holomorphic hull of γ in $\mathbb{C}^N$. By Stolzenberg's theorem $h(\gamma) \backslash \gamma$ is an analytic subset in $\mathbb{C}^N \backslash \gamma$.

Let us show that $h(\gamma) \subset X$. X is a maximal spectrum of functions that are equal to zero on X ([GR65], theorem VII, A18). Take a function f such that $f(X) = 0$. Then $f(h(\gamma)) = 0$ since $\gamma \subset X$ and $h(\gamma)$ is a holomorphic hull of γ in $\mathbb{C}^N$. Thus, $h(\gamma) \subset X$.

It remains to show that $h(\gamma) = h_X(\gamma)$. Any holomorphic function on X is a restriction of holomorphic function on $\mathbb{C}^N$ ([GR65], theorem VII, A18). Therefore, $h_X(\gamma) = h(\gamma) \cap X$. Since $h(\gamma) \subset X$, $h_X(\gamma) = h(\gamma)$. □

1.5.3 Holomorphic convexity of a degenerate object.

In this subsection we give the geometric conditions for the degenerate objects to be holomorphically convex.

Definition 1.5.1. We say that a path or a loop is *simple* if it does not have points of self-intersection.

We need to extend the analytic set, given by Stolzenberg's theorem, to the boundary. In the sequel we need the following corollary from the Stolzenberg's Theorem.

Corollary 1.5.2. *Let* $\gamma_1, \dots, \gamma_n$ *be piece-wise smooth curves, such that* $\gamma_i \cap \gamma_j$ *consists of finite number of points. Suppose that* $h(\gamma) \not\subset \gamma$. *Then there exists an arc* $\alpha \subset \gamma_i$, *such that* $\alpha \subset \partial\left(h(\gamma) \backslash \gamma\right)$, *where* $\gamma = \cup \gamma_i$.

Proof. Let π be a projection of $\gamma_1, \dots, \gamma_n$ to a complex line C. One can choose π so that image of γ has at most finite number of points of transversal self-intersection. The image of γ separate C into several regions U_i.

Below we show that for each of the regions there is a following dichotomy: either $\pi(h(\gamma)) \supset U_i$ or $\pi(h(\gamma)) \cap \operatorname{int}(U_i) = \emptyset$:

First, $\pi\left(h(\gamma) \backslash \gamma\right)$ is open. Therefore, the $\pi(h(\gamma) \backslash \gamma) \cap U_i$ is open. Second, if $w \in \partial \pi (h(\gamma))$, then since $h(\gamma)$ is compact, there exists $z \in h(\gamma)$ such that $\pi(z) = w$. Therefore, $z \in \gamma$. Thus, $\pi(h(\gamma) \backslash \gamma) \cap U_i$ is either empty or coincides with U_i.

Take a point $w \in \pi(\gamma)$ that is not a point of self-intersection and belongs to the boundary of $\pi(h(\gamma))$. The small arc around this point also belongs to $\pi(h(\gamma))$ and does not contain points of self-intersection. The preimage of this arc is the desired arc on γ. □

Theorem 1.5.4 ([Chi89]). *Let* M *be a connected* $(2p-1)$*-dimensional* C^1*-submanifold of a complex manifold* Ω. *Let* A_1, A_2 *be irreducible* p*-dimensional analytic subsets of* $\Omega \backslash M$ *such that the closure of each of them contains* M. *Then either* $A_1 = A_2$ *or* $A_1 \cup M \cup A_2$ *is an analytic subset of* Ω.

Lemma 1.5.2. *A piece-wise real-analytic simple loop* γ *on a leaf* L *of a foliation is either holomorphically convex or is null homologous on* L.

Proof. Suppose γ is not holomorphically convex. Take an analytic arc $\alpha \subset \gamma$ that belongs to the boundary of $h(\gamma)\backslash\gamma$. The existence of such arc follows from Corollary 1.5.2. One can take a neighborhood $U \subset X$ of the arc α, such that

1. $U \cap \gamma = \alpha$;
2. the connected component of $L \cap U$, that contains α, is a submanifold in U;
3. the arc α separates this connected component into two pieces. Let Ω_1, Ω_2 be these pieces.

Let h_1 denote the connected component of $h(\gamma)\backslash\gamma$.

Apply Theorem 1.5.4 to the analytic sets h_1 and Ω_1 and the arc α. The closure of h_1 in U contains α. The closure of Ω_1 also contains α. Therefore, either $h_1 = \Omega_1$ or $h_1 \cup \alpha \cup \Omega_1$ is an analytic subset of U. In the second case $h_1 = \Omega_2$. Thus, $h_1 = \Omega_1$ or $h_1 = \Omega_2$. If two analytic sets coincide locally, then they coincide globally. Therefore, $h(\gamma) \subset L$.

By Maximum Modulus Principle, $\partial h(\gamma) \subset \gamma$. Since γ is a simple curve, it means $h(\gamma)$ is a region on L with the boundary γ, so γ is null homologous on L. □

Theorem 1.5.5. *Let γ be a piece-wise real-analytic loop on the leaf L of foliation $\mathcal{F}_0$ that satisfies additional conditions:*

1. *γ is simple;*
2. *γ is not null homologous on L.*

Then there exists a neighborhood U of γ and a one-dimensional holomorphic family of foliations $\mathcal{F}_\lambda$, such that for $\lambda \in V\backslash R$, all the cycles on the leaves of foliation $\mathcal{F}_\lambda$, close to γ are hyperbolic. Here V is a neighborhood of 0, R is a 1-dimensional real-analytic set, $0 \in R$.

Proof. The theorem is a consequence of Lemmas 1.5.1 and 1.5.2 □

Lemma 1.5.3. *Let γ_1, γ_2 be a pair of piece-wise real-analytic loops on a leaf L of a foliation $\mathcal{F}_0$ such that*

1. *γ_1 and γ_2 are simple loops;*
2. *γ_1 and γ_2 have only one common point;*
3. *γ_1 and γ_2 are not null homologous and are not multiples of the same cycle in the homology group of L.*

Then $\gamma = \gamma_1 \cup \gamma_2$ is holomorphically convex.

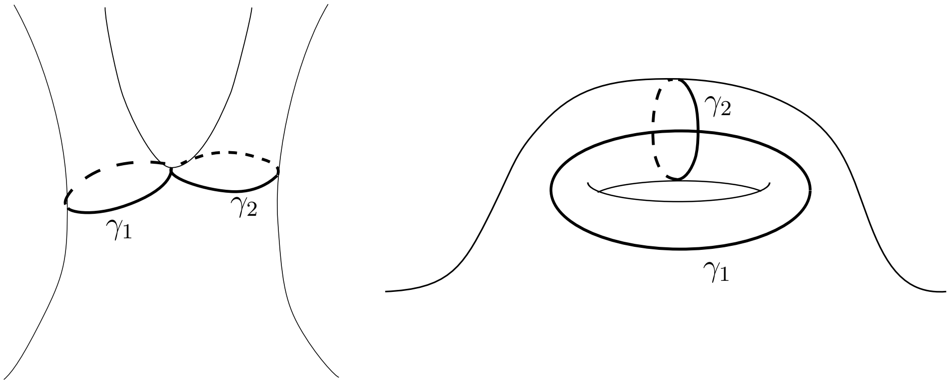


Figure 1.17: A pair of cycles

Proof. Suppose that γ is not holomorphically convex. Then by corollary 1.5.1 $h(\gamma)\backslash\gamma$ is a 1-dimensional analytic subset of $X\backslash\gamma$. By Lemma 1.5.2 there exists an arc $\alpha \subset \gamma_i$ such that $\alpha \subset \partial h(\gamma)$. By the same argument as in the previous lemma, we see that $h(\gamma)$ is a subset of L. Since cycles have only one common point, the boundary of $h(\gamma)$ is either γ_1, γ_2 or $\gamma_1 \cup \gamma_2$. All three cases contradict the hypothesis of the lemma. Thus, γ is holomorphically convex. $\square$

Theorem 1.5.6. *Let γ_1, γ_2 be a pair of piece-wise real-analytic loops on the leaf L of foliation $\mathcal{F}_0$ that satisfy the following additional conditions:*

1. *γ_1 and γ_2 are simple loops;*

2. *γ_1 and γ_2 have only one common point;*

3. *γ_1 and γ_2 are not null homologous and they are not multiples of the same cycle in the homology group of L.*

Then there exists a neighborhood U of $\gamma_1 \cup \gamma_2$ and a family of foliations $\mathcal{F}_\lambda$ such that for all $\lambda \neq 0$ there does not exist a pair of cycles in U close to γ_1, γ_2 correspondingly that lie on one and the same connected component of a leaf of $\mathcal{F}_\lambda$ in U.

Proof. The Theorem is a consequence of Lemmas 1.5.1 and 1.5.3. □

Lemma 1.5.4. *Let γ be a simple piece-wise real-analytic path on a saddle connection S of the foliation $\mathcal{F}$, connecting two different singular points a_1 and a_2. Then γ is holomorphically convex.*

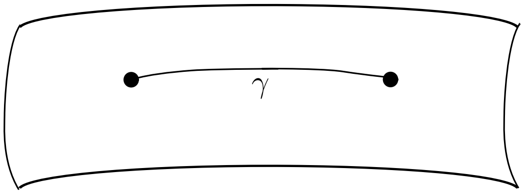


Figure 1.18: A path on a saddle connection

Proof. By the same argument as in Lemma 1.5.2, $h(\gamma) \subset S \cup \{a_1\} \cup \{a_2\}$. By Corollary 1.5.1, $h(\gamma)\backslash\gamma$ is an analytic subset of $X\backslash\gamma$. Therefore, by Maximum Modulus Principle $\partial h(\gamma) \subset \gamma$. But γ is contractible on $S \cup \{a_1\} \cup \{a_2\}$. It cannot bound a region on $S \cup \{a_1\} \cup \{a_2\}$. Therefore, $h(\gamma) = \gamma$. □

Theorem 1.5.7. *Let S be a saddle connection of a foliation $\mathcal{F}_0$. Assume S connects two different singular points a_1 and a_2 of $\mathcal{F}_0$. Let $\gamma \subset L$ be a simple piece-wise real-analytic path, connecting a_1 and a_2. There exists a neighborhood U and a holomorphic family of foliations $\mathcal{F}_\lambda$ such that for all $\lambda \neq 0$ there does not exist a saddle connection S_λ of $\mathcal{F}_\lambda$ with a path on it that connects singular points, and is close to γ.*

Proof. The theorem is a consequence of Lemmas 1.5.1 and 1.5.4. ☐

Lemma 1.5.5. *Let γ be a simple piece-wise real-analytic curve on a homoclinic saddle connection S, that connects a singular point a to itself. Assume that one end of γ belongs to one local component of S, the other end belongs to the other local component. Then γ is holomorphically convex.*

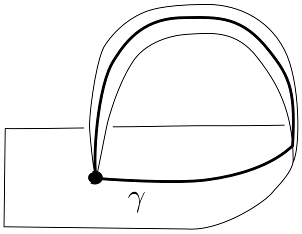


Figure 1.19: A path on a homoclinic saddle connection

Proof. By the same argument as in Lemma 1.5.2, $h(\gamma) \subset S \cup \{a\}$. By Lemma 1.5.1 $h(\gamma)\backslash\gamma$ is an analytic subset of $X\backslash\gamma$. Therefore, by the Maximum Modulus Principle, $\partial\left(h(\gamma)\backslash\gamma\right) \subset \gamma$.

Let $\tilde{S}$ be a surface obtained from $S \cup \{a\}$ by splitting the local components of S at the point a. Let $\pi : \tilde{S} \to S$ be the corresponding projection.

$\pi^{-1}(\gamma)$ is a simple path on $\tilde{S}$. It does not bound a region on $\tilde{S}$. Therefore, its image does not bound a region on S. ☐

Theorem 1.5.8. *Let γ be a simple piece-wise real-analytic curve on a homoclinic saddle connection S, that connects a singular point a to itself. Assume that one end of γ belongs to one local component of S, the other end belongs to the other local component. Then there exists a family of global holomorphic foliations $\mathcal{F}_\lambda$, so that there does not exist a saddle connection S_λ, with a cycle close to γ on it, that connects a singular point to itself.*

Proof. The theorem is a consequence of Lemmas 1.5.1 and 1.5.5. ☐

Lemma 1.5.6. *Let S be a separatrix of the foliation $\mathcal{F}_0$, passing through a singular point a. Let γ be a piece-wise real-analytic loop on S that satisfies additional conditions:*

1. *γ is simple;*

2. *γ passes though the point a;*

3. *γ is not null homologous on $S \cup a$.*

Then γ is holomorphically convex.

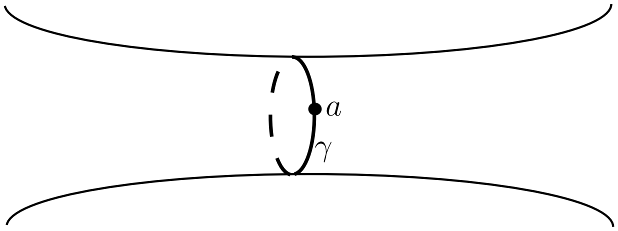


Figure 1.20: A loop on a separatrix.

Proof. The proof is the same as in Lemma 1.5.2. □

Theorem 1.5.9. *Let S be a separatrix of the foliation $\mathcal{F}_0$, passing through a singular point a. Let γ be a piece-wise real-analytic loop on S that satisfies additional conditions:*

1. *γ is simple;*

2. *γ passes though the point a;*

3. *γ is not null homologous on $S \cup a$.*

Then there exists a neighborhood U of γ and a family of global holomorphic foliations $\mathcal{F}_\lambda$ such that for all $\lambda \neq 0$, there does not exist a cycle close to γ that lies on a separatrix S_λ of the foliation $\mathcal{F}_\lambda$.

Proof. The theorem is a consequence of Lemmas 1.5.1 and 1.5.6. □

Lemma 1.5.7. *Let M_1 and M_2 be strongly invariant manifolds of hyperbolic singular points a_1 and a_2 of a foliation $\mathcal{F}_0$. Let p be a point of non-transversal intersection of M_1 and M_2. Let $\gamma = \gamma_1 \cup \gamma_2$ be a simple piece-wise real analytic path, where $\gamma_1 \subset M_1$ connects a_1 and p; $\gamma_2 \subset M_2$ connects p and a_2. Then γ is holomorphically convex.*

Proof. Suppose $h(\gamma)\backslash\gamma \neq \emptyset$. By corollary 1.5.2 there is an arc $\alpha \subset \partial\left(h(\gamma)\backslash\gamma\right)$. Consider a holomorphic curve C that passes through α. Without loss of generality we can assume $\alpha \subset \gamma_1$. By the same arguments as in the previous lemmas $h(\gamma)\backslash\gamma \subset C$. However, $C \subset M_1$, therefore, $\partial\left(h(\gamma)\backslash\gamma\right) \subset \gamma_1$. But γ_1 is a path, so it does not bound a region on C. Therefore, $h(\gamma) = \gamma$. □

Theorem 1.5.10. *Let M_1 and M_2 be strongly invariant manifolds of hyperbolic singular points a_1 and a_2 of a foliation $\mathcal{F}_0$. Let p be a point of non-transversal intersection of M_1 and M_2. Let $\gamma = \gamma_1 \cup \gamma_2$ be a simple piece-wise real analytic path, where $\gamma_1 \subset M_1$ connects a_1 and p; $\gamma_2 \subset M_2$ connects p and a_2. Assume there exists a leaf L of the foliation $\mathcal{F}_0$ such that $\gamma\backslash\left(M_1^{loc} \cup M_2^{loc}\right) \subset L$. Then there exists a neighborhood U of γ and a holomorphic family of foliations $\mathcal{F}_\lambda$, such that M_1^λ and M_2^λ, obtained by continuing local invariant manifolds along γ, intersect transversally in a neighborhood of the point p.*

Proof. The theorem is a consequence of Lemmas 1.5.1 and 1.5.7 □

Lemma 1.5.8. *Let M_1 and M_2 be strongly invariant manifolds of a hyperbolic singular point a. Let p be a point of non-transversal intersection of M_1 and M_2. Let $\gamma = \gamma_1 \cup \gamma_2$ be a simple piece-wise real analytic loop, where $\gamma_1 \subset M_1$ connects a and p; $\gamma_2 \subset M_2$ connects p and a.Assume that $M_1^{loc} \cap M_2^{loc} = a$. Then γ is holomorphically convex.*

Proof. The same as in the previous lemma. □

Theorem 1.5.11. *Let M_1 and M_2 be strongly invariant manifolds of a hyperbolic singular point a of a foliation $\mathcal{F}_0$. Assume $M_1^{loc} \cap M_2^{loc} = a$. Let p be a point of non-transversal intersection of M_1 and M_2. Let $\gamma = \gamma_1 \cup \gamma_2$ be a simple piece-wise real analytic loop, where $\gamma_1 \subset M_1$ connects a and p; $\gamma_2 \subset M_2$ connects p and a. Assume there exists a leaf L of the foliation $\mathcal{F}_0$ such that $\gamma\backslash\left(M_1^{loc} \cup M_2^{loc}\right) \subset L$. Then there exists a neighborhood U of γ and a holomorphic family of foliations $\mathcal{F}_\lambda$ so that the strongly invariant manifolds*

M_1^λ, M_2^λ, obtained by continuing local invariant manifolds along γ, intersect transversally in a neighborhood of a point p.

Proof. The theorem is a consequence of Lemmas 1.5.1 and 1.5.8. □

1.6 Simultaneous elimination of degeneracies.

1.6.1 Landis-Petrovskii's lemma.

The idea is to encode degeneracies by countably many objects. To give a feeling of the method used, we first prove a version of the Landis-Petrovskii's Lemma [LP55] that we need in the sequel.

Lemma 1.6.1. *For a holomorphic* 1*-dimensional (singular) foliation $\mathcal{F}$ of a Stein manifold X there exists not more than a countable number of isolated complex cycles on the leaves of the foliation.*

Proof. Since the manifold X is Stein, it can be embedded into $\mathbb{C}^N$. Take a cycle γ on a leaf L.

Fix coordinates $(z_1, \dots, z_N)$ in $\mathbb{C}^N$. Let $C_1, \dots, C_N$ be the coordinate lines,

$$C_i = \{z_1, \dots, = \hat{z}_i = \dots = z_N = 0\}.$$

Suppose that L does not belong to the hypersurface $\{z_i = c\}$ for any $c \in \mathbb{C}$. By perturbing γ on the leaf L one can assume that there exists a small neighborhood $U \supset \gamma$ so that $\pi_i|_U$ is a biholomorphism to the image (here $\pi_i : \mathbb{C}^N \to C_i,\ \pi_i(z) = z_i$ is the projection). Then one can perturb γ inside U so that $\pi_i(\gamma)$ becomes a piece-wise linear curve with rational vertices.

Definition 1.6.1. We will say that the cycle γ' *lies over* the piece-wise linear curve g' if there exist a representative of γ' and its neighborhood U', such that U' is projected biholomorphically to its image and the representative is projected to g'. Note, that any cycle lies over countably many piece-wise linear curves.

Take one of the vertices of $\pi_i(\gamma)$, say with coordinate $z_i = c$. The hypersurface $\{z_i = c\}$ intersects X by $(k-1)$-dimensional variety, such that for any cycle γ', lying over $\pi_i(\gamma)$, it is transversal to the foliation in a neighborhood of $\gamma' \cap \{z_i = c\}$. The holonomy map along γ is well-define in some neighborhood of the intersection $\{z_i = c\} \cap \gamma$. The holonomy map does not have any other fixed points in some smaller neighborhood. Thus, each cycle that projects to the same piece-wise linear curve gives a neighborhood on the hyperplane $\{z_i = c\} \subset \mathbb{C}^N$, so that two neighborhoods for two different cycles do not intersect each other. Therefore, there exists not more than countably many limit cycles that project to the same curve. Since there are only countably many curves, there are not more than countably many limit cycles.

$\square$

Landis-Petrovskii's Lemma implies that once all non-isolated cycles are eliminated, all leaves except for countably many are homeomorphic to disks.

1.6.2 Simultaneous elimination of non-isolated cycles.

If there are non-isolated cycles on the leaves of a foliation $\mathcal{F}$, then the number of the cycles is obviously uncountable. However, the strategy described above can be applied. Our idea is to catch the degenerations by a countable number of holonomy maps.

Since X is Stein, it can be embedded into $\mathbb{C}^N$.

We can restrict ourselves to the foliations without leaves that belong to the hypersurfaces $\{z_N = c\},\ c \in \mathbb{C}$. The set of such foliations is open and dense.

We describe the holonomy maps that catch all the cycles for all foliations.

We introduce the following notations:

- $\mathcal{A}$ is a countable, everywhere dense subset in the set of holomorphic foliations;

- $\mathcal{G}$ is the set of all closed piecewise-linear curves with rational vertices on

  $$\{z_1 = \cdots = z_{N-1} = 0\},$$

  with one marked vertex.

- Let $\tau_q = \{z_n = q\} \cap X$, where $q \in \mathbb{Q} + i\mathbb{Q}$.

  Let $\mathcal{Q}_q$ be a countable everywhere dense set on τ_q.

  Let $\mathcal{Q} = \bigsqcup \mathcal{Q}_q$.

Let $\mathbf{z} = (z_1, \ldots, z_{N-1})$, $u = z_N$.

Consider a 4-tuple $\alpha = (\mathcal{F}, g, \mathbf{z}, r) \in (\mathcal{A}, \mathcal{G}, \mathcal{Q}_q, \mathbb{Q})$, such that q is the marked point of g. We require that the holonomy map for the foliation $\mathcal{F}$ in the point $\mathbf{z}$ along g is well-defined in a neighborhood of $\mathbf{z}$ on the transversal section τ_q and has the radius of convergence greater than r. Let Δ_α be the germ of this holonomy. One can consider the germ of the holonomy map along the lifting of g, starting at $\mathbf{z}$, for foliations close to $\mathcal{F}$. Therefore, we think of Δ_α as of function of two variables: a foliation close to $\mathcal{F}$, and a point on the transversal section τ_q.

Below we fix a specific representative of Δ_α. We use the same notation for the specific representative as for the germ.

Let V_α be the connected component, containing $\mathcal{F}$, of the set of foliations, for which the holonomy map along g in the point $\mathbf{z}$ is well-defined and has radius of convergence greater than r. The domain of definition of Δ_α is

$$\{(\mathcal{F}', \mathbf{z}')|\ \mathcal{F}' \in V_\alpha,\ |\mathbf{z}' - \mathbf{z}| < r\}.$$

Note, that V_α is open.

From this point we consider fixed representatives, rather than germs.

Lemma 1.6.2. *Every complex cycle corresponds to a fixed point of $\Delta_\alpha(\mathcal{F}', \cdot)$ for some α and $\mathcal{F}' \in V_\alpha$.*

Proof. Let γ be a complex cycle on a leaf L of a foliation $\mathcal{F}$. One can perturb γ on L so that it projects to some $g \in \mathcal{G}$. Let $u(g)$ be one of the vertices of the projection, and let $\mathbf{z} \in \gamma$ be the preimage of $u(g)$. Consider the holonomy map along γ in a neighborhood of $\mathbf{z}$ in the transversal section $C = \{u = u(g)\}$. Take a point $\mathbf{z}_1 \in \mathcal{Q}$ such that $|\mathbf{z} - \mathbf{z}_1| < r_{\mathbf{z}}(\mathcal{F})/4$ where $r_{\mathbf{z}}(\mathcal{F})$ is a radius of convergence of the holonomy map in the point $\mathbf{z}$ along γ for the foliation $\mathcal{F}$. Note, that $r_{\mathbf{z}_1}(\mathcal{F}) > r_{\mathbf{z}}(\mathcal{F})/2$. One can take $\mathcal{F}_1$ close to $\mathcal{F}$ so that $r_{\mathbf{z}_1}(\mathcal{F}_1) > r_{\mathbf{z}}(\mathcal{F})/2$. Denote by $\alpha = (\mathcal{F}_1, g, \mathbf{z}_1, r)$, where $r \in \mathbb{Q}, r_{\mathbf{z}}(\mathcal{F})/4 < r < r_{\mathbf{z}}(\mathcal{F})/2$. Then $r < r_{\mathbf{z}_1}(\mathcal{F}_1)$. Also, $\mathcal{F} \in V_\alpha$, because $r_{\mathbf{z}_1}(\mathcal{F}) > r$. Since $r > r_{\mathbf{z}}(\mathcal{F})/4$, the point z belongs to the domain of definition of $\Delta_\alpha(\mathcal{F}_1, \cdot)$. □

Lemma 1.6.3. *Fix Δ_α. The set $D_\alpha \subset V_\alpha$ of foliations $\mathcal{F}$ such that $\Delta_\alpha(\mathcal{F}, \cdot)$ has a non-hyperbolic fixed point, so that the corresponding cycle γ satisfies additional conditions:*

1. *γ is simple,*

2. *γ is null homologous on the leaf;*

is closed and nowhere dense in V_α.

Proof. We prove that by a finite number of steps, we can perturb the foliation $\mathcal{F}$ so that $\Delta_\alpha(\widetilde{\mathcal{F}}, \cdot)$ has isolated fixed points only in the domain of definition discussed above. Assume that A is the set of fixed points of $\Delta_\alpha(\mathcal{F}, \cdot)$. Let A be k-dimensional. As we show in the appendix, one can associate multiplicity $m(A)$ to the analytic set A. Take a point z that is a generic point of a k-dimensional stratum A_i. By Theorem 1.5.5, there exists a neighborhood of z and a foliation $\widetilde{\mathcal{F}}$, arbitrary close to $\mathcal{F}$, such that the holonomy map of $\widetilde{\mathcal{F}}$ along γ has isolated fixed points only in this neighborhood.

This perturbation destroys the component A_i. Therefore, it either decreases the dimension of A, or it decreases the multiplicity $m(A)$ (see Lemma 1.7.3).

Therefore, after a finite number of steps, only isolated cycles are left. By the Theorem 1.5.5, they can be turned into hyperbolic by a finite number of steps as well. □

Corollary 1.6.1. *The complement of D_α in the set of all foliations contains an open every where dense set.*

Theorem 1.6.1. *There exists a residual set $\mathcal{R}_1$ in the space of 1-dimensional singular holomorphic foliations, so that for any foliation $\mathcal{F} \in \mathcal{R}_1$ all non-trivial cycles γ on the leaves of $\mathcal{F}$ are isolated.*

Proof. By intersecting the open every where dense sets from the corollary above, we get the residual set $\mathcal{R}'$, so that for every foliation $\mathcal{F} \in \mathcal{R}'$ all cycles, satisfying additional conditions:

1. γ is simple;
2. γ is null homologous on the leaf;

are hyperbolic and, therefore, isolated. Suppose L is a leaf of a foliation $\mathcal{F} \in \mathcal{R}'$ that has a non-isolated cycle. Notice that if $H^1(L) = 0$, then L is contractible and there are no non-trivial cycles at all. Therefore, there exists a simple homologically non-trivial cycle on L. It follows, that this cycle is isolated. But by Landis-Petrovskii's lemma the number of isolated cycles is countable. Therefore, there could be only countably many leaves with non-isolated cycles. But this contradicts the following lemma:

Lemma 1.6.4 ([Il72])**.** *Let γ_n be a sequence of contractible loops on the leaves of a 1-dimensional singular holomorphic foliation of a complex manifold, suppose $\gamma_n \to \gamma$, where γ is a loop on a leaf. Then γ is contractible.*

Indeed, there are uncountably many non-isolated cycles in a neighborhood of one non-isolated cycle. Therefore, most of them should be trivial. But then the original cycle is trivial as well.

Take $\mathcal{R}_1 = \mathcal{R}'$.

□

1.6.3 Simultaneous splitting of cycles to different leaves.

Theorem 1.6.2. *There exists a residual set $\mathcal{R}_2$ in the space of 1-dimensional singular holomorphic foliations so that all non-separatrix leaves of foliation $\mathcal{F} \in \mathcal{R}_2$ are contractible or homeomorphic to a cylinder.*

Proof. Take $\beta = (\mathcal{F}, g_1, g_2, \mathbf{z}) \in (\mathcal{A}, \mathcal{G}, \mathcal{G}, \mathcal{Q}_q)$ so that

- g_1 and g_2 have the same marked vertex, $u(g_1) = u(g_2) = q$;
- holonomy maps for the foliation $\mathcal{F}$ in the point $\mathbf{z}$ along g_1 and g_2 are well defined in a neighborhood of $\mathbf{z}$ on $\{u = u(g_1)\}$.

  Let Δ_{β_1} and Δ_{β_2} be the holonomy maps along g_1 and g_2 correspondingly.

□

We can fix the common domain of definition of Δ_{β_1}, Δ_{β_2} to be the intersection of domains of definition of Δ_{β_1} and Δ_{β_2}.

Lemma 1.6.5. *Every pair of cycles that lie on the same leaf of* 1*-dimensional singular holomorphic foliation corresponds to a common fixed point of Δ_{β_1} and Δ_{β_2} for some β.*

Proof. The proof is the same as in Lemma 1.6.2. □

Lemma 1.6.6. *Fix β. The set D_{β_1,β_2} of foliations $\mathcal{F}$, so that $\Delta_{\beta_1}(\mathcal{F}, \mathbf{z})$ and $\Delta_{\beta_2}(\mathcal{F}, \mathbf{z})$ have a common fixed point, so that the corresponding cycles γ_1 and γ_2 satisfy additional conditions:*

1. *γ_1, γ_2 are simple;*
2. *γ_1, γ_2 have one point of intersection;*
3. *γ_1, γ_2 are null homologous on the leaf; they are not multiples of the same cycle in the homology group of the leaf; and the leaf is not a separatrix.*

is closed and nowhere dense in the common domain of definition of Δ_{β_1}, Δ_{β_2}.

Proof. The proof is the same as for lemma 1.6.3. □

Corollary 1.6.2. *For every pair* Δ_{β_1} *and* Δ_{β_2} *there exist an open everywhere dense set in the complement to the* D_{β_1,β_2} *in the set of all foliations.*

Intersecting open every-where dense sets from the corollary, and intersecting with R_1, we get a residual set $\mathcal{R}'$, so that for any $\mathcal{F} \in \mathcal{R}'$ all cycles on the leaves of $\mathcal{F}$ are isolated, and there is no pair of cycles γ_1, γ_2 on a leaf of $\mathcal{F}$ that satisfy additional conditions (1)-(3). As it was shown in [F06], if the Riemann surface is not contractible and not homeomorphic to the cylinder, then there exists a pair of cycles that satisfy conditions (1)-(3). Therefore, if L is not a separatrix, then L is contractible or homeomorphic to a cylinder.

Take $\mathcal{R}_2 = \mathcal{R}'$.

1.6.4 Simultaneous elimination of non-transversal intersections of strongly invariant manifolds.

Since X is a Stein manifold, it can be embedded into $\mathbb{C}^N$.

We fix the countable set of data $\alpha = (\mathcal{F}, a_1, M_1, a_2, M_2, g, z_1, r)$.

- $\mathcal{F} \in \mathcal{A}$, where $\mathcal{A}$ is a countable every-where dense set of foliations;

Foliations with hyperbolic singular points only form a residual set [Ch04]. Therefore, we can assume that all singular point for all the foliations $\mathcal{F} \in \mathcal{A}$ are hyperbolic.

- a_1, a_2 are hyperbolic singular points of $\mathcal{F}$;

- M_1, M_2 are strongly invariant manifolds of a_1 and a_2 correspondingly;

We associate the maximal radius r_i to the singular point a_i.

Definition 1.6.2. The radius r_i is the maximal radius, such that M_i is transversal to $\partial U_r(a_i)$ for all $r < r_i$.

Not that maximal radius is a lower semicontinuous function on the space of foliations. Let $\pi : X \to C$ be the projection to $C = \{z_1 = \cdots = z_{N-1} = 0\}$, $\pi(x_1, \ldots, x_N) = x_N$.

- $g \subset C$ is a piecewise linear curve with rational vertices. Let u_1, u_2 be the starting and the ending points of g correspondingly. We require that $u_1 \in \pi(U_{r_1}(a_1))$, $u_2 \in \pi(U_{r_2}(a_2))$;
- $z_1 \in \mathcal{Q}_q$, where $\mathcal{Q}_q$ is an every where dense set on the transversal section $\tau_1 = \{z_n = u_1 = q\} \cap X$ in $U_{r_1}(a_1)$;

We require that there is a well-defined lift of g to the leaf L of the foliation $\mathcal{F}$, that starts from a point z_1. The lift is denoted by γ. Let z_2 be the lift u_2. We require that $z_2 \in U_{r_2}(a_2)$

Let $\tau_2 = \{z_N = u_2\} \cap X$.

There is a well-defined germ $\Delta : \tau_1 \to \tau_2$ of the holonomy map along γ in the point z_1.

As before, we think of Δ as a function of two variables: a foliation $\mathcal{G}$, close to $\mathcal{F}$, and a point on the transversal section τ_1.

- $r \in \mathcal{Q}_+$. We require that
  1. r is less than radius of convergence of Δ.
  2. The disk $D_r(z_1)$ on the transversal section τ_1 of the radius r_1 with the center z_1 is compactly contained in $U_{r_1}(a_1)$.
  3. $\Delta(D_r(z_1))$ is compactly contained in $U_{r_2}(a_2)$.

We fix a representative Δ_α of Δ. Below we describe the neighborhood U_α of $\mathcal{F}$. $\mathcal{G}$ belongs to U_α if

1. there is a holomorphic family of foliations $\mathcal{F}_\lambda$, so that $\mathcal{F}_0 = \mathcal{F}$, $\mathcal{F}_1 = \mathcal{G}$; for all $\lambda \in D_1$ there are unique hyperbolic singular points $a_1^\lambda \in U_{r_1/2}(a_1)$ and $a_2^\lambda \in U_{r_2/2}(a_2)$ of the foliation $\mathcal{F}_\lambda$;

   Let a_1', a_2' be singular points of $\mathcal{G}$, obtained via holomorphic continuation. Let M_1', M_2' be the corresponding strongly invariant manifolds. Let r_1', r_2' be the maximal radii for (a_1', M_1'), (a_2', M_2').

2. $z_1 \in U_{r_1'}(a_1')$, $z_2' \in U_{r_2'}(a_2')$, where z_2' is the lift of u_2 along g for $\mathcal{G}$.

3. $D_r(z_1)$ is compactly contained in $U_{r_1'}(a_1')$.

4. $\Delta(\mathcal{G}, D_r(z_1))$ is compactly contained in $U_{r_2'}(a_2')$.

The domain of definition of Δ_α is $U_\alpha \times D_r(z_1)$.

Lemma 1.6.7. *For any α, the set $D_\alpha \subset U_\alpha$ of foliations $\mathcal{G} \subset U_\alpha$, for which there exists a leaf L such that*

1. *the lift of u_1 to L is in $U_{r_1}(a_1)$, the lift of u_2 to L is in $U_{r_2}(a_2)$;*

2. *the lift of g belongs to the strongly invariant manifold M_1' of the singular point a_1' of $\mathcal{G}$ (a_1' is a holomorphic continuation of a_1);*

3. *the lift of u_2 belongs to the strongly invariant manifold M_2' of a singular point a_2' (a_2' is a holomorphic continuation of a_2);*

4. *the lift of u_2 is a point of a non-transversal intersection of M_1' and M_2'.*

is a closed and nowhere dense set.

Proof. The proof follows from the local Theorem 1.5.10 in the same way as in Lemma 1.6.3. □

Corollary 1.6.3. *The space of foliations, such that they do not have a non-transversal intersection of invariant manifolds, that correspond to data α, contains an open and everywhere dense set.*

Theorem 1.6.3. *There exists a residual set $\mathcal{R}_3$ in the space of 1-dimensional holomorphic singular foliations such that*

1. *all singular points are hyperbolic;*
2. *strongly invariant manifolds of different singular points intersect transversally.*

Proof. Take $\mathcal{R}_3$ to be an intersection of open everywhere dense sets from the previous corollary. Take $\mathcal{F} \in \mathcal{R}_3$. Suppose there is a non-transversal intersection of invariant manifolds for $\mathcal{F}$. Take a point p of a non-transversal intersection of strongly invariant manifolds M_1 and M_2. The point $p \in L$, where L is a leaf of foliation $\mathcal{F}$. There is a point $z_2 \in L$, so that

1. $z_2 \in U_{r_2/2}(a_2)$;
2. z_2 is a point of a non-transversal intersection of strongly invariant manifolds;
3. $\pi(z_2)$ is rational.

Take $z_1 \in L$, $z_1 \in U_{r_1/2}(a_1)$. Take a path γ on L that lies over a piece-wise linear path g.

$u_1 = \pi(z_1)$, $u_2 = \pi(z_2)$.

There exists $\mathcal{G} \in \mathcal{A}$, so that

1. there are singular points a_1', a_2' of $\mathcal{G}$, that are holomorphic continuations of a_1, a_2.
2. Strongly invariant manifolds M_1', M_2' of a_1' and a_2' are holomorphic continuations of M_1, M_2.
3. $z_1 \in U_{r_1'}(a_1')$, $z_2 \in U_{r_2'}(a_2')$.

One can take $z_1' \in \mathcal{Q}_{u_1}$, at a small enough distance $r/2$ from z_1, so that

1. $D_r(z_1')$ is compactly contained in $U_{r_1}(a_1)$ and $U_{r_1'}(a_1')$;
2. $\Delta(\mathcal{F}, D_r(z_1'))$ is compactly contained in $U_{r_1}(a_1)$;
3. $\Delta(\mathcal{G}, D_r(z_1'))$ is compactly contained in $U_{r_1'}(a_1')$

$\mathcal{F}$ belongs to U_β, where $\beta = (\mathcal{G}, a_1', M_1', a_2', M_2, g, z_1', r)$. Point z_1 belongs to the domain of definition of $\Delta_\beta(\mathcal{F}, \cdot)$ and corresponds to a point of a non-transversal intersection, which is impossible. This completes the proof. □

Theorem 1.6.4. *There exists a residual set in the space of holomorphic* 1*-dimensional singular foliations such that*

1. *all singular points are hyperbolic.*
2. *Let a_1 be a hyperbolic singular point of the foliation. Let M_1 and M_2 be strongly invariant manifolds of the point a_1, such that $M_1^{loc} \cap M_2^{loc} = a_1$. Then M_1 and M_2 intersect transversally everywhere.*

Proof. The proof is the same as in the Theorem 1.6.3. □

1.6.5 Simultaneous elimination of saddle connections

Theorem 1.6.5. *There exists a residual set $\mathcal{R}_3$ in the space of holomorphic singular* 1*-dimensional foliations such that*

1. *all the singular points are hyperbolic;*
2. *there are no saddle connections between two different singular points;*
3. *there are no homoclinic saddle connections.*

Proof. The proof can be carried as a simplified version of the proof of Theorem 1.6.3. □

Theorem 1.6.6. *There exists a residual set $\mathcal{R}_4$ in the space of 1-dimensional holomorphic singular foliations, so that there are no saddle connections, and all separatrices are cylinders.*

Proof. The construction of holonomy maps that catch all the degeneracies is the same as for strongly invariant manifolds. By intersecting the open-every where dense sets that correspond to these holonomy maps we get a residual set for which there and no saddle connections and there are no cycles that satisfy some additional conditions. The way to get rid of additional conditions is the same as in Section 1.6.3. Consult [F06] for further consideration. □

Proof of the theorem 1.1.1. $\mathcal{R}' = \mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_4 \cap \mathcal{R}_5$ is a residual set, satisfying all the hypothesis of the theorem.

Since $\mathcal{R}' \subset \mathcal{R}_1$, for any $\mathcal{F} \in \mathcal{R}'$, all complex cycles are isolated.

Since $\mathcal{R}' \subset \mathcal{R}_2 \cap \mathcal{R}_4 \cap \mathcal{R}_5$ all leaves are either contractible or cylinders. But a generator γ of a fundamental group of a cylinder L is

1. not null homologous;

2. can be chosen to be simple.

Therefore, by Lemma 1.6.3 it is hyperbolic. The rest of the cycles on L are multiples of γ. Therefore, they are hyperbolic as well. □

Proof of the theorem 1.1.2. The residual set $\mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3 \cap \mathcal{R}_4 \cap \mathcal{R}_5$ satisfies all the hypothesis of the Theorem 1.1.2. □

1.7 Appendix

1.7.1 Multiplicity

We consider analytic subsets A of a polydisk $\bar{D}^n$, i.e. we assume that A is an analytic subset of some neighborhood of D^n. Suppose that A is given by a system of n equations

$$f_1 = \cdots = f_n = 0$$

Assume that A is k-dimensional. We want to define the multiplicity of A which does not increase under perturbations.

Lemma 1.7.1. *There are only finitely many strata of maximal dimension.*

Proof. The number of strata is locally finite [Chi89]. Since A is an analytic subset of $\bar{D}^n$, it is globally finite. □

Let $A_1, \dots, A_m$ be the strata of maximal dimension.

Take a smooth point $z \in A_i$. Consider a transversal section T to A_i at the point z. Let $\tilde{f}_1, \dots, \tilde{f}_n$ be the restriction of $f_1, \dots, f_n$ to T. The point z is an isolated solution of the system

$$\tilde{f}_1 = \cdots = \tilde{f}_n = 0.$$

Definition 1.7.1. Let z be an isolated point of a system of equations

$$\tilde{f}_1 = \cdots = \tilde{f}_n = 0$$

defined in $n-k$-dimensional polydisk D^{n-k}. The multiplicity $m(z)$ of a point z is the dimension of

$$\mathcal{O}_{D^{n-k},z} / < \tilde{f}_1, \dots, \tilde{f}_n >,$$

where $\mathcal{O}_{D^{n-k},z}$ is the local ring of $z \in D^{n-k}$, i.e. functions, regular in a neighborhood of $z \in D^{n-k}$; $< \tilde{f}_1, \dots, \tilde{f}_n >$ is the ideal in $\mathcal{O}_{D^{n-k},z}$ generated by $\tilde{f}_1, \dots, \tilde{f}_n$.

Lemma 1.7.2. *The multiplicity does not increase under perturbations, i.e. if $z'_1, \dots, z'_m$ are isolated solutions of a perturbed system in a neighborhood of a point z, then*

$$\sum_{i=1}^{m} m(z'_i) \leq m(z).$$

Proof. In [AGV85] it is proved for $k = 0$. In general case the proof goes the same way. ☐

Definition 1.7.2. The multiplicity of $z \in A_i$ is the multiplicity the point z as an isolated solution of $\tilde{f}_1 = \dots = \tilde{f}_n = 0$.

It is easy to see that multiplicity does not depend on the choice of a generic point and a transversal section T.

Definition 1.7.3. The multiplicity of a stratum A_i is the multiplicity of a generic point. The multiplicity of A is the sum of multiplicities of A_i.

Lemma 1.7.3. *The multiplicity of A does not increase under perturbations, i.e. let $A'_1, \dots A'_{m'}$ be strata of a perturbed system, then*

$$\sum_{i=1}^{m'} m(A'_i) \leq m(A).$$

Proof. Let $T_1, \dots, T_m$ be transversal sections to A_i's at generic points. Every A'_i intersect at least one of the sections $T_1, \dots, T_m$. One can also assume that T_i's meet A_i's transversally. On each transversal section the result follows from the Lemma 1.7.2.

☐

Chapter 2

Critical Locus for Complex Hénon Maps

2.1 Introduction. Foliations.

Hénon mappings are polynomial automorphisms of $\mathbb{C}^2$ of the form

$$f_a(x) = \begin{pmatrix} p(x) - ay \\ x \end{pmatrix}.$$

Both real and holomorphic versions of these maps have been extensively studied. Some of the sources of the fundamental results about Hénon mappings are [BLS93],[BS91a],[BS91b], [BS92], [BS98a], [BS98b], [BS99], [HOV94],[HOV95] [FM89] [FS92].

We study quadratic Hénon mappings:

$$p(x) = x^2 + c.$$

Inspired by one-dimensional dynamics, it is useful to consider the following dynamically defined sets:

$$U_a^+ = \{(x, y) : \lim_{n\to\infty} |f_a^n(x, y)| = \infty\};$$

$$K_a^+ = \mathbb{C}^2 \backslash U_a^+;\ J_a^+ = \partial K_a^+;$$

$$U_a^- = \{(x,y) : \lim_{n\to\infty} |f_a^{-n}(x,y)| = \infty\}.$$

$$K_a^- = \mathbb{C}^2 \backslash U_a^-;\ J_a^- = \partial K_a^-;$$

$$J_a = J_a^+ \cap J_a^-.$$

We restrict our attention to the set $U_a^+ \cap U_a^-$

The growth rate of the orbit under the iteration of f_a is measured by the function G_a^+:

$$G_a^+(x,y) = \lim_{n\to\infty} \frac{\log^+ |f_a^n(x,y)|}{2^n}.$$

Lemma 2.1.1 ([HOV94]). *$G_a^+(x,y)$ is a well-defined function for all $(x,y) \in \mathbb{C}^2$. Moreover,*

$$G_a^+(x,y) > 0 \Leftrightarrow (x,y) \in U_a^+.$$

G_a^+ is pluriharmonic on U_a^+.

The growth rate of the orbit under the iteration of f_a^{-1} is measured by the function G_a^- :

$$G_a^-(x,y) = \lim_{n\to\infty} \frac{\log^+ |f_a^{-n}(x,y)|}{2^n} + \log|a|$$

We add $\log|a|$ to have a limit as $a \to 0$. The analog of Lemma 2.1.1 holds for G_a^-

Since G_a^+, G_a^- are pluriharmonic functions, their level sets are foliated by Riemann surfaces. We denote the foliation of U_a^+ by $\mathcal{F}_a^+$, the foliation of U_a^- by $\mathcal{F}_a^-$.

The dynamical description for the foliations is the following:

Lemma 2.1.2 ([BS98a]). *The leaves of $\mathcal{F}_a^+$ are superstable manifolds, i.e. if z_1, z_2 belong to the same leaf, then $d(f_a^n(z_1), f_a^n(z_2)) \to 0$ super exponentially. If z_1, z_2 do not belong to the same leaf, then $d(f_a^n(z_1), f_a^n(z_2)) \not\to 0$.*

The same is true for $\mathcal{F}_a^-$.

One would want to think of these foliations as providing coordinates in $U_a^+ \cap U_a^-$. However, for all Hénon mappings there is a codimension one subvariety of tangencies between $\mathcal{F}_a^+$ and $\mathcal{F}_a^-$[BS98a].

Definition 2.1.1. *Critical locus $\mathcal{C}_a$ is the set of tangencies between foliations $\mathcal{F}_a^+$ and $\mathcal{F}_a^-$.*

So the critical locus is the set of heteroclinic tangencies between the superstable and superunstable manifolds.

The foliations $\mathcal{F}_a^+, \mathcal{F}_a^-$ can also be characterized in terms of Böttcher coordinates.

For a one-dimensional map $x \to p(x)$ there is the Böttcher coordinate that conjugates the dynamics at the basin of infinity to $z \to z^2$.

There are analogs of the Böttcher coordinate for Hénon mappings. There are maps $\varphi_{a,+}$ and $\varphi_{a,-}$ that semiconjugate the map f_a and the map f_a^{-1} to $z \to z^2$ and $z \to z^2/a$ correspondingly. These functions were constructed in [HOV94]. We recall the definitions of these functions and list their properties in Section 2.4.

The level sets of $\varphi_{a,+}$ and $\varphi_{a,-}$ define $\mathcal{F}_a^+$ and $\mathcal{F}_a^-$ in some small domains and can be propagated by dynamics to all of U_a^+ and U_a^- correspondingly.

2.2 Topological model for the critical locus.

In [LR] Lyubich and Robertson gave the description of the critical locus for Hénon mappings $(x, y) \to (p(x) - ay, x)$, where $p(x)$ is a hyperbolic polynomial with connected Julia set, a is sufficiently small. They showed that for each critical point c of p there is a component of the critical locus that is asymptotic to the line $y = c$. Each component of the critical locus is an iterate of these ones, and each is a punctured disk.

They used critical locus to show that a pair of quadratic Hénon maps of the studied type, taken along with the natural foliations, gives a rigid object. In the sense that if a conjugacy between two Hénon maps sends the natural foliations of the first map to the natural foliations of the second map then the two Hénon maps are conjugated by a holomorphic or antiholomorphic affine map.

We provide the topological model for Hénon mappings $(x, y) \to (x^2 + c - ay, x)$, where $x^2 + c$ has disconnected Julia set, and a is sufficiently small.

The truncated spheres serve as building blocks for the critical locus. Consider a sphere S and a pair of disjoint Cantor sets $\Sigma, \Omega \subset S$. The elements of Σ, Ω can be parametrized by one-sided infinite sequences of $0, 1$'s. Denote by $\sigma_\alpha \in \Sigma$, $\omega_\alpha \in \Omega$ elements parametrized by a sequence α.

Let α_n be a n-string of 0's and 1's. For each α_n, $n \in \mathbb{N} \cup \{0\}$, take a disk $V_{\alpha_n} \subset S \setminus (\Sigma \cup \Omega)$ with the smooth boundary. (Let V denote the disk that corresponds to an empty sequence.) We require that V_{α_n} are disjoint. Moreover, V_{α_n} converge to σ_α, where α_n is the string of the first n elements of α.

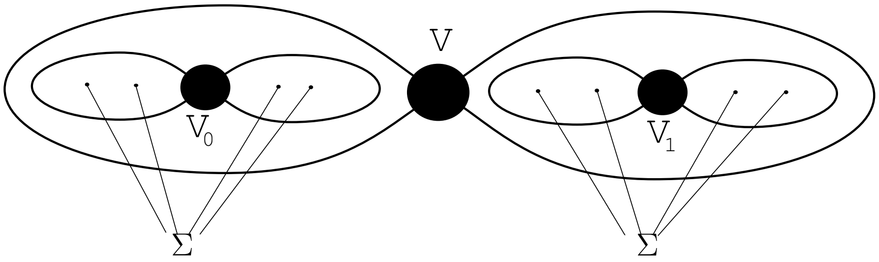


Figure 2.1: The geometry of a truncated sphere

U_{α_n} play the same role for Ω as V_{α_n} for Σ.

We assume that there is a fixed homeomorphism $\tilde{h}$ between the boundaries of V_{α_n} and U_{α_n}.

Let $p \in S$ be a point.

We say that $S\backslash\left(\Sigma\cup\Omega\cup\sum_{\alpha_n}[U_{\alpha_n}\cup V_{\alpha_n}]\cup p\right)$ is a truncated sphere.

Let us prove that all truncated spheres are homeomorphic one to another.

First, we prove a preliminary lemma.

Lemma 2.2.1. *Let D_1 and D_2 be unit disks with smooth boundaries. Let $\Sigma_1 \subset D_1$, $\Sigma_2 \subset D_2$ be Cantor sets. Then $D_1\backslash\Sigma_1$ and $D_2\backslash\Sigma_2$ are homeomorphic.*

Proof. Since Σ_1, Σ_2 are Cantor sets, they are homeomorphic. Denote by $h : \Sigma_1 \to \Sigma_2$ the homemorphism.

The elements of Cantor sets can be parametrized by sequences of 0 and 1's and we can choose parametrizations that respects h. Denote the elements of Σ_1 and Σ_2, that are parametrized by α, by σ^1_α and σ^2_α correspondingly.

$$h(\sigma^1_\alpha) = \sigma^2_\alpha.$$

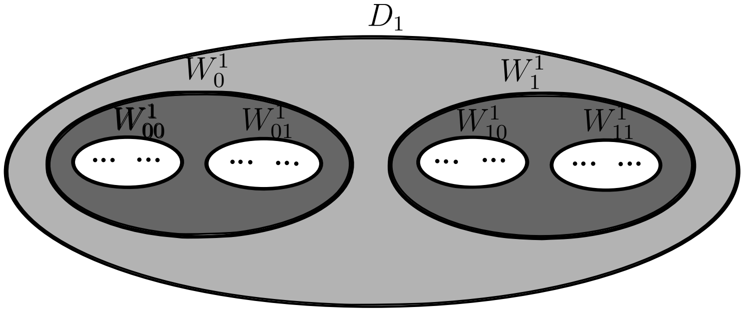


Figure 2.2: The partition of a disk, with a Cantor set removed, into pairs of pants

There exist open sets $W^1_{\alpha_n} \subset D_1$, parametrized by n-strings α_n of 0 and 1's, so that $W^1_{\alpha_n}$ contains all points σ_α, such that the string of the first n-digits of the sequence α is α_n.

We can take $W^1_{\alpha_n}$'s with smooth boundaries. And assume that

$$W^1_{\alpha_{n+1}}, W^1_{\alpha'_{n+1}} \Subset W^1_{\alpha_n},$$

where α_{n+1}, α'_{n+1} are obtained from α_n by adding correspondingly 0 and 1 at the end.

Repeat the same procedure for Σ_2, and denote the corresponding sets by $W^2_{\alpha_n}$.

Fix some homeomorphism $h : \partial D_1 \to \partial D_2$.

Obviously, one can extend h to a homeomorphism

$$h : D_1 \backslash (W^1_0 \cup W^1_1) \to D_2 \backslash (W^2_0 \cup W^2_1)$$

so that it restricts to homeomorphisms

$$h : \partial W^1_0 \to \partial W^2_0$$

$$h : \partial W^1_1 \to \partial W^2_1$$

Now we can extend it to the next pair of pants and so on. In the end we get a homeomorphism $h : D_1 \backslash \Sigma_1 \to D_2 \backslash \Sigma_2$. □

Lemma 2.2.2. *The truncated sphere is a well-defined object up to a homeomorphism.*

Proof. Let S_1 and S_2 be two truncated spheres. We add indices 1 and 2 to Cantor sets and disks, used in the definition truncated spheres.

Let us cut S_1 into two disks, so that the first disk D_1 contains Σ_1 and all disks $V^1_{\alpha_n}$. The second disk contains Ω_1 and all disks $U^1_{\alpha_n}$.

Do the same procedure with S_2. Denote the corresponding disk by D_2.

Now we prove that there exists a homeomorphism h

$$h : D_1 \backslash \left(\Sigma_1 \cup V^1_{\alpha_n} \right) \to D_2 \backslash \left(\Sigma_2 \cup V^2_{\alpha_n} \right)$$

Using the notations from the previous lemma one can choose W_{α_n}, so that they contain all sets V_{α_m}, such that

1. $m \geq n$,

2. α_n is the first n-string of α_m.

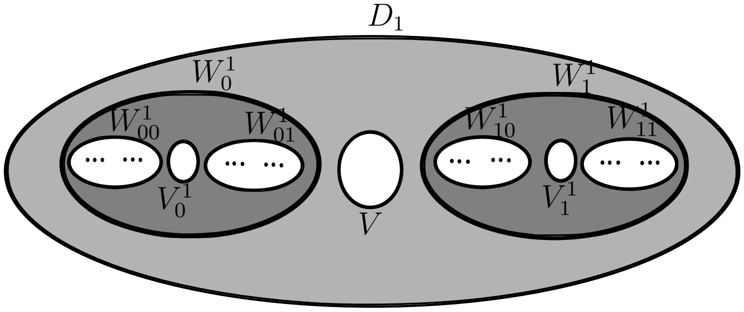


Figure 2.3: The partition of a half of a truncated sphere into pairs of pants with a hole

On each step we extend the homeomorphism from the boundary to a pair of pants with a hole.

We do the same procedure for the part of truncated spheres left. We create a homeomorphism, which respects $\tilde{h}$.

$\square$

Theorem 2.2.1. *Suppose $x^2 + c$ has disconnected Julia set. There exists δ, such that $\forall |a| < \delta$ the critical locus of the map*

$$f_a \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + c - ay \\ x \end{pmatrix}$$

is smooth, and has the following topological model: take countably many truncated spheres S_n, $n \in \mathbb{Z}$, and glue the boundary of V_{α_n} on S_k to the boundary of U_{α_n} of S_{n+k} using the homeomorphism $\tilde{h}$. Map f_a acts on the critical locus by sending S_n to S_{n+1}.

Lemma 2.2.3. *The topological model of the critical locus is well-defined up to a homeomorphism.*

Proof. It is easy to see that the homeomorphism from Lemma 2.2.2 commutes with the gluing. $\square$

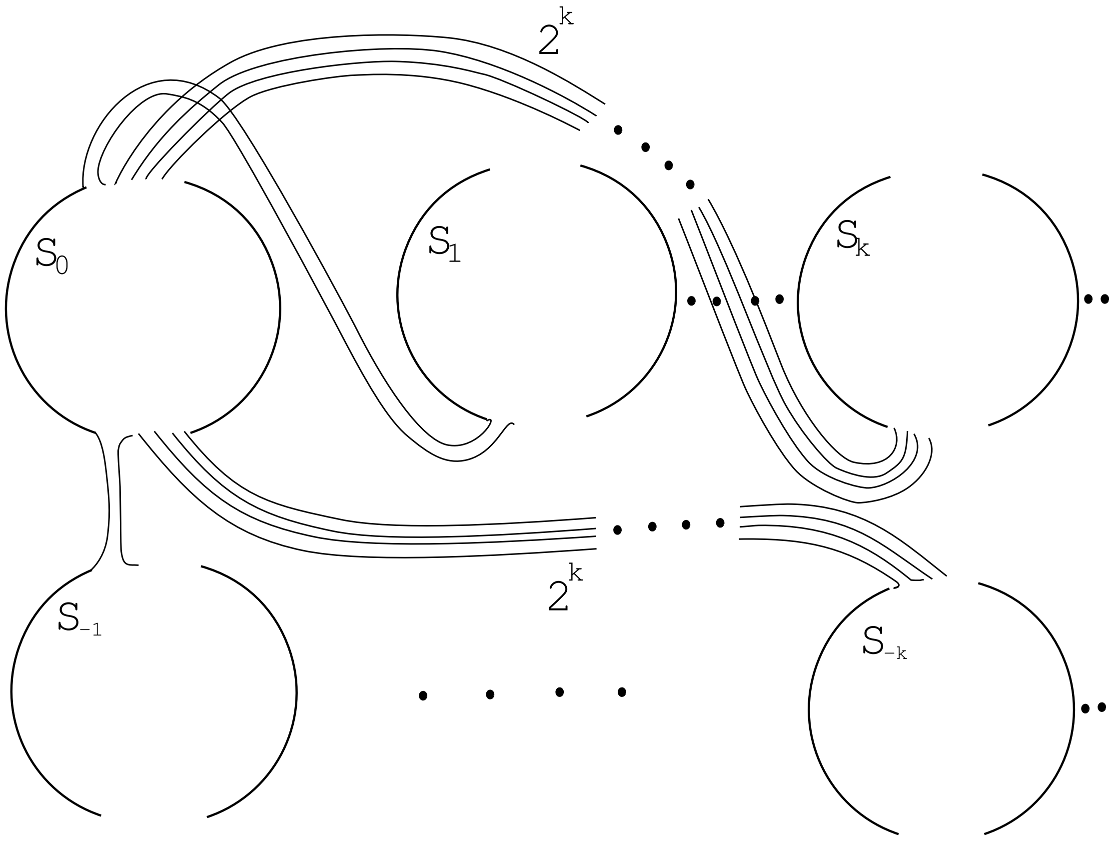


Figure 2.4: The topological model.

2.3 Strategy of description.

When $a = 0$, the Hénon mapping reduces to

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x^2 + c \\ x \end{pmatrix}$$

which is a map $x \rightarrow p(x)$ acting on the curve $x = p(y)$.

As $a \rightarrow 0$ the map degenerates, but the foliations and the Green functions persist and become easy to analyze.

In Section 2.4 we study $\varphi_{a,+}$ and $\varphi_{a,-}$, paying extra attention to the degeneration as $a \rightarrow 0$.

Section 2.5 is devoted to Green's functions. We prove that G_a^+ and G_a^- depend continuously on x, y and the parameter a.

In Section 2.6 we describe the foliations $\mathcal{F}_a^+$ and $\mathcal{F}_a^-$ and the critical locus $\mathcal{C}$ in terms of $\varphi_{a,+}$ and $\varphi_{a,-}$. We also calculate the critical locus in the degenerate case. As a deviates from zero, we carefully describe the perturbation. The latter is done in several steps:

First, we choose appropriate values r and α and describe the critical locus in the domain

$$\Omega = \{G_a^+ \leq r\} \cap \{|y| \leq \alpha\} \cap \{|p(y) - x| \geq |a|\alpha\}.$$

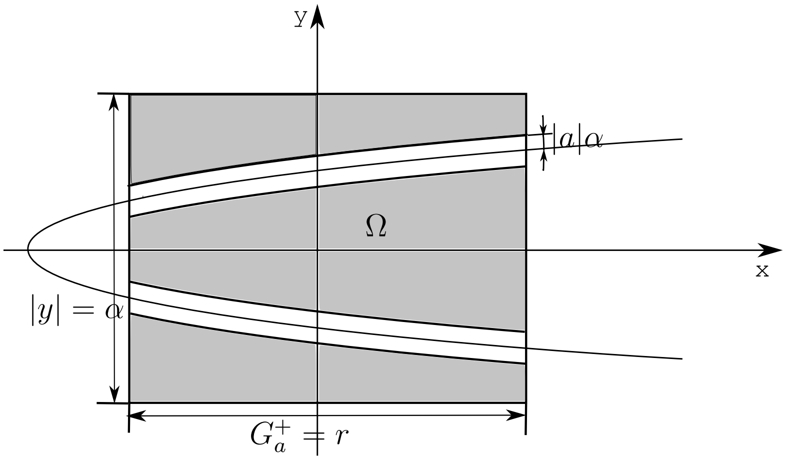


Figure 2.5: Domain Ω

We choose r that lies between the critical point level and critical value level:

$$G_p(0) < r < G_p(c),$$

where G_p is the Green function of polynomial p.

We choose α such that $G_a^+|_{\{|x|>\alpha, |x|>|y|\}} > r$. Moreover, in Section 2.9 we choose α such that leaves of foliation $\mathcal{F}_a^-$ in Ω form a family of parabolas.

In Section 2.10 we give a description of the foliation $\mathcal{F}_a^+$ in

$$\{G_a^+ \leq r\} \cap \{|y| \leq \alpha\}.$$

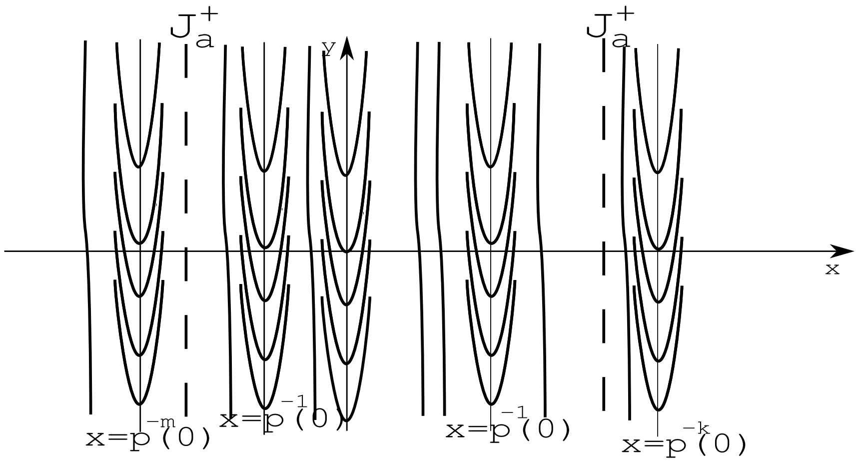


Figure 2.6: The foliation $\mathcal{F}_a^+$

We show that the local leaves of $\mathcal{F}_a^+$ are either vertical-like or parabolas. There is exactly one family of vertical parabolas (see Fig.2.3) corresponding to each line $x = \xi_k$, where $p^{\circ k}(\xi_k) = 0$.

The critical locus in Ω is the set of tangencies of a family of horizontal parabolas with vertical-like leaves and vertical parabolic leaves. In Section 2.11 we show that for each family of parabolas there is exactly one handle.

In Section 2.7 we show that foliations $\mathcal{F}_a^+$ and $\mathcal{F}_a^-$ extend holomorphically to $x = \infty$. We show that a point $(0, \infty)$ belongs to the extension and calculate the tangent line to the critical locus at this point. This gives us a description of the critical locus in

$$\{|y| < \varepsilon\} \cap \{|x| > \alpha\}.$$

In Section 2.12 we show that the critical locus can be extended up to $a|\alpha|$-neighborhood of parabola $p(y) - x = \text{const}$ along $y = 0$.

In Section 2.13 we combine the results from the previous sections to get the description of the fundamental domain of the certain component of the critical locus. We rule out ghost components. We also do a dynamical regluing of the fundamental domain of the critical locus to obtain a description in terms of truncated spheres.

2.4 Functions $\varphi_{a,+}$ and $\varphi_{a,-}$.

In this section we construct functions $\varphi_{a,+}$ and $\varphi_{a,-}$. In their description we follow [HOV94]. In description of the degeneration of $\varphi_{a,+}$ and $\varphi_{a,-}$ as $a \to 0$, we follow [LR].

2.4.1 Large scale behavior of the Hénon map

The study of Hénon mappings usually begins with introduction of domains V_+ and V_- which are invariant under the action of Hénon map:

$$f_a(V_+) \subset V_+, \quad f_a^{-1}(V_-) \subset V_-.$$

Moreover, one requires that every point that has unbounded forward orbit eventually enters V_+ and every point that has unbounded backward orbit enters V_-.

Fix R, $0 < r < 1$, and choose α so that

$$\frac{|c|}{|y^2|} + \frac{R+1}{|y|} < r \tag{2.4.1}$$

$$|p(y)| > (2R+1)|y| \tag{2.4.2}$$

for all $|y| \geq \alpha$.

Consider the following partition of $\mathbb{C}^2$:

$$V_+ = \{(x,y) :\ |x| > |y|, |x| > |\alpha|\};$$

$$V_- = \{(x,y) :\ |y| > |x|, |y| > \alpha\};$$

$$W = \{(x,y) :\ |x| \leq \alpha, |y| \leq \alpha\}.$$

Lemma 2.4.1. *For $a \in D_R$, $f_a(V_+) \subset V_+$.*

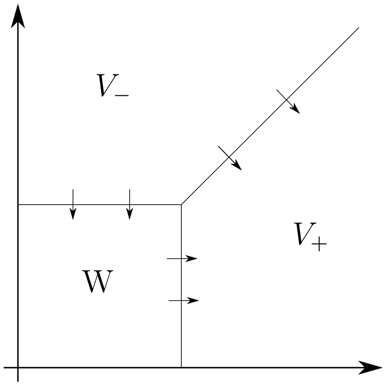


Figure 2.7: The crude picture of the dynamics of f_a

Proof. $|p(x) - ay| \geq |p(x)| + |a||y| \geq (2R + 1 - R)|y| \geq |y|$

$|p(x) - ay| \geq (R + 1)|y| \geq \alpha$ ☐

Let

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = f_a^n \begin{pmatrix} x \\ y \end{pmatrix}.$$

Lemma 2.4.2. $U_a^+ = \bigcup_n f_a^{-n}(V_+)$;

Proof. Let $f_a^n(x) \to \infty$. For some n, $f_a^n(x) \in V_+$ or $f_a^n(x) \in V_-$.

Suppose that for all $n \geq n_0$ $(x_n, y_n) \in V_-$. Note that $|y_{n+1}| = |x_n| \leq |y_n|$. Sequence $\{|y_n|\}$ is decreasing. Contradiction. ☐

Lemma 2.4.3. *For $a \in D_R$ $f_a^{-1}(V_-) \subset V_-$.*

Proof. $\frac{|p(y)-x|}{|a|} \geq \frac{|2R+1||y|-|x|}{R} \geq |y|$

$\frac{|p(y)-x|}{|a|} \geq |y| \geq |\alpha|$ ☐

Lemma 2.4.4. $U_a^- = \cup_n f_a^n(V_-)$.

Proof. The proof is the same as in the previous lemma. ☐

2.4.2 Function $\varphi_{a,+}$

The function $\varphi_{a,+}$ is constructed as the limit

$$\varphi_{a,+}(x,y) = \lim_{n\to\infty} |x_n|^{\frac{1}{2^n}}.$$

with the appropriate choice of the branch of root. We are particularly interested in this function in V_+. Sense is made of the above limit using the telescopic formula

$$\varphi_{a,+} = \lim_{n\to\infty} x \exp\left(\frac{1}{2}\log\frac{x_1}{x^2} + \dots + \frac{1}{2^n}\log\frac{x_n}{x_{n-1}^2} + \dots\right). \tag{2.1}$$

Lemma 2.4.5. *The function $\varphi_{a,+}$, defined by formula (2.1), is well-defined and holomorphic for all $(x,y) \in V_+$ and all $a \in D_R$. Moreover, there exists B so that*

$$B^{-1} < \left|\frac{\varphi_{a,+}}{x}\right| < B.$$

Proof. Let $s_n^+ = \frac{x_n}{x_{n-1}^2} - 1$. By property (2.4.1), $|s_n^+| < r$.
$\log \frac{x_n}{x_{n-1}^2} = \log(1+s_n^+)$ is calculated using the principle branch of log. Therefore, the series

$$\frac{1}{2}\log\frac{x_1}{x^2} + \dots + \frac{1}{2^n}\log\frac{x_n}{x_{n-1}^2} + \dots \tag{2.2}$$

converges absolutely and uniformly. Since $|\log\frac{x_n}{x_{n-1}^n}| < -\frac{\log(1-r)}{2^n}$, the infinite sum (2.2) is no bigger than $-\log B = -\log(1-r)$.

The final claim follows immediately from the expression (2.1) and the bound derived for the series (2.2). □

We show that $\varphi_{a,+} \sim x$ as $x \to \infty$ in Section 2.7.

Let $\mathcal{D}_{n,+} = \{(x,y,a)|\ f_a^n(x,y,a) \in V_+, a \in D_R\}$

Lemma 2.4.6. *The function $\varphi_{a,+}^{2^n}$ extends to a holomorphic function on $\mathcal{D}_{n,+}$ given by $\varphi_{a,+}^{2^n} = \varphi_{a,+} \circ f_a^n$. Moreover, $\varphi_{0,+}^{2^n}(x,y) = b_p^{2^n}(x)$.*

Proof. The function $\varphi_{a,+}^{2^n}$ is holomorphic by definition.

As $a \to 0$ the Hénon mappings degenerate to a one-dimensional map $x \to p(x)$, acting on $y = p(x)$. Therefore, $\varphi_{0,+}(x, y) = b_p(x)$ on V_+, and $\varphi_{0,+}^{2^n}(x, y) = b_p^{2^n}(x)$ if $f_0^n(x, y) \in V_+$. $\square$

$$K_0^+ = J_0^+ = J_p \times \mathbb{C},$$

$$U_0^+ = U_p \times \mathbb{C},$$

where J_p is the Julia set for the one-dimensional map $x \to p(x)$; U_p is the set of points, whose orbits escape to ∞ under the map $x \to p(x)$.

2.4.3 Function $\varphi_{a,-}$

We start by working the leading terms of y_{-n} as a polynomial in y and as a polynomial in $\frac{1}{a}$. The following notation simplifies the statement of the result:

$\sigma_k = 1 + 2 + \cdots + 2^{k-1}$ for $k \geq 1$ and $\sigma_k = 0$ for $k \leq 0$.

By an easy induction we get:

Lemma 2.4.7. *The leading term of $y_{-n}(x, y, a)$ considered as a polynomial in y is y^{2^n}/a^{σ_n}. The leading term of y_{-n} considered as a polynomial in $\frac{1}{a}$ is $\frac{1}{a^{\sigma_n}}(p(y) - x)^{2^{n-1}}$.*

We define the function $\varphi_{a,-}$ as a limit

$$\varphi_{a,-} = \lim_{n\to\infty} (y_{-n} \circ a^{\sigma_n})^{\frac{1}{2^n}} \tag{2.3}$$

with an appropriate choice of branch of root. Note that the factor a^{σ_n} is chosen, so that the leading term of $y_n \cdot a^{\sigma_n}$, as a polynomial in y, is y^{2^n}.

Let $D_R^* = D_R \backslash \{0\}$.

Sense is made of the limit in $V_- \times D_R^*$ using the telescopic formula:

$$\varphi_{a,-}(x,y)=\lim_{n\to\infty}\exp\left(\frac{1}{2}\log\frac{ay_{-1}}{y^2}+\frac{1}{2^2}\log\frac{ay_{-2}}{y_{-1}^2}+\dots\right) \tag{2.4}$$

Lemma 2.4.8. *The function $\varphi_{a,-}$, defined by formula (2.4) is well-defined and holomorphic on $V_- \times D_R^*$. Moreover, there exists B so that*

$$B^{-1}<\left|\frac{\varphi_{a,-}}{y}\right|<B.$$

Proof. Let $s_n^- = \frac{c-x_{-(n-1)}}{y^2_{-(n-1)}}$

Note that $|s_n^-|<r$ for $(x,y)\in V_-$, $a\in D_R^*$.

We evaluate $\frac{ay_{-n}}{y^2_{-(n-1)}}=\log(1+s_n^-)$ using the principle branch of log.

Since $\log(1+s_n^-)\le -\log(1-r)$, the series

$$\frac{1}{2}\log\frac{ay_{-1}}{y^2}+\frac{1}{2^2}\frac{ay_{-2}}{y_{-1}^2}+\dots \tag{2.5}$$

converges uniformly and absolutely to a holomorphic function bounded by $-\log(1-r)$. Letting $B=(r-1)^{-1}$, the last statement of the Lemma follows. □

We show that $\varphi_{a,-}\sim y$ as $y\to\infty$ in Section 2.7.

The next lemma states that one can extend $\varphi_{a,-}$ to a holomorphic function on $V_-\times D_R$.

Lemma 2.4.9. $\varphi_{a,-}=(p(y)-x)^{\frac{1}{2}}+ah(x,y,a)$ *for some holomorphic function h on $V_-\times D_R$.*

Proof. By Lemma 2.4.7 the leading term of y_{-n} as a polynomial in $\frac{1}{a}$ is $\frac{1}{a^{\sigma n}}(p(y)-x)^{2^{n-1}}$. Since $x_{-n}=y_{-(n-1)}$. The leading term of x_{-n} as a polynomial in $\frac{1}{a}$ is the leading term of $y_{-(n-1)}$.

Recall $s_n^-=\frac{c-x_{-(n-1)}}{y^2_{-(n-1)}}$. It follows that s_n^- is a polynomial in a and it vanishes in a to the order $2\sigma_{n-1}-\sigma_{n-2}$. Hence the series (2.5) takes the form

$$\frac{1}{2}\log\frac{ay_{-1}}{y^2}+ag(x,y,a),$$

where $g(x, y, a)$ is a holomorphic function on $V_+ \times D_R$.

By (2.4), $\varphi_{a,-}(x, y) = y \exp\left(\frac{1}{2} \log \frac{ay_1}{y^2})\right) \exp(ag(x, y, a))$. The conclusion follows. $\square$

Let

$$C(p) = \{p(y) - x = 0\}$$

Domain $f_a(V_-)$ swells to $\mathbb{C}^2 \backslash C(p)$ as $a \to 0$.

Lemma 2.4.10. *For $(x, y) \in \mathbb{C}^2 \backslash C(p)$ we have $(x, y) \in f_a(V_-)$ for all sufficiently small a. Moreover, for all $K \Subset \mathbb{C}^2 \backslash C(p) \Rightarrow K \Subset f_a(V_-)$ for all small enough a.*

Proof. Both statements follow from

$$f_a(V_-) = \{|p(y) - x| \geq |a|\alpha, |p(y) - x| \geq |a||y|\}.$$

$\square$

$$\mathcal{D}_{-,n} = \{(x, y, a)|\ (x, y) \in f_a^n(V_-), a \neq 0 \text{ or } (x, y) \in \mathbb{C}^2 \backslash C(p), a = 0\}.$$

Lemma 2.4.11. *The function $\varphi_{a,-}^{2^n}(x, y)$ extends from a holomorphic function on $V_- \times D_R$ to $\mathcal{D}_{-,n}$, by letting*

1. $\varphi_{a,-}^{2^n} = a^{\sigma_n} \varphi_{a,-} \circ f_a^{-n}$ *for* $a \neq 0$;

2. $\varphi_{0,-}^{2^n}(x, y) = (p(y) - x)^{2^{n-1}}$.

Proof. If $a \neq 0$, then we can extend the function $\varphi_{a,-}$ to be holomorphic by defining $\varphi_{a,-}^{2^n} = a^{\sigma_n} \varphi_{a,-} \circ f_a^{-n}$. This agrees with $\varphi_{a,-}^{2^n}$ on V_-.

By Lemma 2.4.9

$$\varphi_{a,-}^{2^n} = a^{\sigma_n} \varphi_{a,-} \circ f_a^{-n} = a^{\sigma_n} (p(y_{-n}) - x_{-n})^{\frac{1}{2}} + a^{\sigma_n + 1} h(x_{-n}, y_{-n}, a) \tag{2.6}$$

By Lemma 2.4.7 the leading term of y_{-n}, as a polynomial in $\frac{1}{a}$, is $\frac{(p(y)-x)^{2^n}}{a^{\sigma}}$. Since $x_{-n} = y_{-(n-1)}$, the leading term of x_{-n} as a polynomial in $\frac{1}{a}$ is $\frac{(p(y)-x)^{2^{n-2}}}{a^{\sigma_{n-1}}}$. Thus $a^{\sigma_n}(p(y_{-n}) - x_{-n})^{\frac{1}{2}}$ is a polynomial in x, y, a and the only term free in a is $(p(y)-x)^{2^{n-1}}$.

By the same argument the function $a^{\sigma_n} h(x_{-n}, y_{-n}, a)$ is holomorphic in (x, y, a) on $V_+ \times D_R$. Therefore, $a^{\sigma_n+1} h(x_{-n}, y_{-n}, a)$ vanishes in a.

Thus, $\varphi_{a,-}^{2^n} = (p(y) - x)^{2^{n-1}} + ah_1(x, y, a)$. □

We denote K_0^- and J_0^- to be $C(p)$. We set $U_0^- = \mathbb{C}^2 \backslash J_0^-$. The previous two statements justify these notations.

2.5 Green's functions

The next two lemmas are from [HOV94]. We provide the proof of Lemma 2.5.1 to make the thesis self-contained.

Lemma 2.5.1. *The function*

$$G_a^+(x, y) = \lim_{n\to\infty} \log^+ |f_a^n(x, y)| \tag{2.7}$$

is well-defined in $\mathbb{C}^2 \times D_R$. *It satisfies the functional equation*

$$G_a^+(f_a) = 2G_a^+. \tag{2.8}$$

Moreover, it is pluriharmonic in U_a^+.

Proof. We define, the function $G_a^+ = \log|\varphi_{a,+}|$ on V_+. We use the functional equation (2.8) to extend it to U_a^+. We also set $G_a^+(x, y) = 0$ for $(x, y) \in J_a^+$. G_a^+, defined this way, is pluriharmonic in V_+, since it is a logarithm of a holomorphic non-zero function $\varphi_{a,+}$. It is pluriharmonic in U_a^+, since for each n, on $f_a^{-n}(V_+)$, G_a^+ is a pull-back of a pluriharmonic function by a holomorphic change of coordinates.

Moreover, notice that the function defined this way satisfies (2.7) and there is a unique function that satisfies (2.7). □

Lemma 2.5.2. *The function*

$$G_a^-(x,y) = \lim_{n\to\infty} |f_a^{-n}(x,y)| + \log|a| \tag{2.9}$$

is well-defined in $\mathbb{C}^2 \times D_R^*$. *It satisfies the functional equation*

$$G_a^-(x,y) \circ f_a^- = 2G_a^- - \log|a|. \tag{2.10}$$

Moreover, it is pluriharmonic on U_a^-.

Equation (2.10) is sometimes more conveniently written

$$(G_a^- - \log|a|) \circ f_a^{-1} = 2(G_a^- - \log|a|).$$

We set

$$G_0^-(x,y) = \log|\varphi_{0,-}(x,y)| = \begin{cases} \frac{1}{2}\log|p(y)-x| & \text{for } (x,y) \notin C(p); \\ -\infty & \text{for } (x,y) \in C(p). \end{cases} \tag{2.11}$$

Hubbard & Oberste-Vorth proved that the Green's functions are continuous when f_a is non-degenerate and the same argument gives continuity in x, y and a for G_a^+ when $a = 0$. Lyubich-Robertson [LR] extend this to G_a^- when $a = 0$.

Lemma 2.5.3 ([LR])**.** *The functions* $G_a^+(x,y)$ *and* $G_a^-(x,y)$ *are continuous in* x, y *and* a *for* $a \in D_R$.

Proof. It follows by the same argument as is used in [HOV94] except in the case of G_a^- when $a = 0$. For $(x', y') \notin C(p)$ the continuity of G_a^- at (x', y') and $a = 0$ follows from Lemma 2.4.11. For $(x', y') \in C(p)$ more work is required. If we restrict G_a^- to the slice

$a = 0$, then we have shown continuity, so we will assume for most of the rest of this proof that $a \neq 0$ (so f_a^{-1} is defined).

Let us fix $M > 0$, and find a neighborhood U of (x', y') so that for all $(x, y) \in U$, $G_a^-(x, y) < -M$

If $(x, y) \in J_a^-$, then it is enough to require that $|a| < e^{-M}$.

If $(x, y) \in U_a^-$, then $f_a^{-n}(x, y) \in V_-$ for all $n > n_0$.

If $f_a^{-n}(x, y) \in V_-$, then by Lemma 2.4.8 $B^{-1} < \left|\frac{\varphi_{a,-}(x_{-n}, y_{-n})}{y_{-n}}\right| < B$.

Therefore, $G_a^-(x, y) < \frac{1}{2}\log B + \frac{1}{2^n}\log|a^{\sigma_n} y_{-n}|$.

$$y_{-n} = \frac{1}{a}\left(p(y_{-(n-1)}) - y_{-(n-2)}\right) \text{ for } n \geq 2. \tag{2.12}$$

$$y_0 = y,\ y_{-1} = \frac{p(y) - x}{a}$$

We wish to estimate $a^{\sigma_n} y_{-n}$ in terms of y_0 and y_{-n}.

It is convenient to introduce a new variable $z_{-n} = a^{\sigma_n} y_{-n}$ and a notation $p\,(\,x\,,\ y\,)\ =\ y^2\,p\,(\,\frac{x}{y}\,)$.

In these new notations the recurrence relation (2.12) takes form

$$z_{-n} = p(z_{-(n-1)}, a^{\sigma_{n-1}}) - a^{\sigma_n - \sigma_{n-2} - 1} z_{n-2}, \tag{2.13}$$

where $z_0 = y$, $z_{-1} = p(y) - x$.

$|z_{-n}| \leq 2\max(|p(z_{-(n-1)}|, |a|^{\sigma_{n-1}}), |a|^{2^n + 2^{n-1} - 1}|z_{n-2}|)$

Using the estimate $p(x, y) \leq C\max(x^2, a^2)$, where C is a constant, we get

$|z_{-n}| \leq 2\max(|z^2_{-(n-1)}|, |a|^{2\sigma_{n-1}}, |a|^{2^n + 2^{n-1} - 1}|z_{n-2}|)$

Consider a neighborhood

$$U(\varepsilon) = \{(x, y, a)|\ |p(y) - x| < \varepsilon^2, |a| < \min(2C\varepsilon^2, \frac{1}{2}), |a||y| < C\varepsilon^2\} \tag{2.14}$$

By induction one can show that if $(x, y, a) \in U(\varepsilon)$, then $z_{-n} < (2C)^{\sigma_n}\varepsilon^{2^n}$.

Therefore, for $(x, y, a) \in U(\varepsilon)$

$$G_a^-(x,y,a) < \log(|2C|\varepsilon) \tag{2.15}$$

Therefore, for small enough ε, $G_a^-(x,y,a) < -M$.

Intersecting $U(\varepsilon)$ with the e^{-M}-neighborhood in a-variable, we get the desired neighborhood.

□

2.6 Description of the foliations and the critical locus in terms of $\varphi_{a,+}$ and $\varphi_{a,-}$

Note that

$$G_a^+(x,y) = \log|\varphi_{a,+}(x,y)|;$$

Therefore, the foliation $\varphi_{a,+}$ = const on V_+ is exactly $\mathcal{F}_a^+$. It can be propagated by dynamics to the rest of U_a^+.

The same way,

$$G_a^-(x,y) = \log|\varphi_{a,-}(x,y)|.$$

The foliation $\varphi_{a,-}$ = const on V_- is $\mathcal{F}_a^-$ and it propagates by dynamics to U_a^-

$$\mathcal{U}^\pm = \{(x,y,a)|\ a \in D_R, (x,y) \in U_a^\pm\}$$

Below we show that the critical locus is defined by a global holomorphic form on $\mathcal{U}^+ \cap \mathcal{U}^-$. Therefore, it is a proper analytic subset of $\mathcal{U}_+ \cap \mathcal{U}_-$.

The forms $d\log\varphi_{a,+}$ and $d\log\varphi_{a,+}$ are well-defined and holomorphic in $\mathcal{U}^+$ and $\mathcal{U}^-$ correspondingly.

Critical locus $\mathcal{C}$ is given by the zeroes of the form

$$w(x,y,a)dx \wedge dy \wedge da = d\log\varphi_{a,+} \wedge d\log\varphi_{a,-} \wedge da$$

that is holomorphic in $\mathcal{U}_+ \cap \mathcal{U}_-$.

Let $\mathcal{C}_{a_0} = \mathcal{C} \cap \{a = a_0\}$.

We recall a definition of a proper analytic subset:

Definition 2.6.1. *A is a proper analytic subset of a complex manifold M, if for every point $x \in M$, there exist a neighborhood U and a set of functions $f_1, \ldots, f_n$ so that*

$$f_1 = \cdots = f_n = 0$$

define A in U.

Lemma 2.6.1. *$\mathcal{C}$ is a proper analytic subset of $\mathcal{U}_+ \cap \mathcal{U}_-$.*

Proof. $\mathcal{C}$ is defined by the zeroes of the form that is holomorphic in $\mathcal{U}_+ \cap \mathcal{U}_-$. □

2.6.1 Degenerate critical locus.

As $a \to 0$ the Hénon mapping degenerates to $(x,y) \to (p(x),x)$. The functions $\varphi_{a,+}^{2^n}$ and $\varphi_{a,-}^{2^n}$ extend as holomorphic functions to $\mathcal{D}_{n,+}$ and $\mathcal{D}_{n,-}$. The level sets $\{\varphi_{a,+}^{2^n} = \text{const}\}$ define the foliation $\mathcal{F}_a^+$ for $a \in D_R^*$. Therefore, the natural analog of the foliation $\mathcal{F}_0^+$ is the foliation defined the level sets of the function $\{\varphi_{0,+}^{2^n} = \text{const}\}$.

The foliation we obtain this way is degenerate. It has double leaves. Below we explain what it means for the foliation to have double leaf.

Suppose that a foliation $\mathcal{F}$ in $\Omega \subset \mathbb{C}^2$ is defined by the level sets of a function φ. Suppose that in a neighborhood of each point (x,y) one can choose local coordinates (u,t), so that $\varphi = u^n$.

Definition 2.6.2. *We say that a leaf L of the foliation $\mathcal{F}$ is double if in a neighborhood of a point $(x,y) \in L$, $\varphi = u^2$.*

Note 2.6.1. *The definition depends on the defining function φ and does not depend on the choice of the local parameter u, nor on a point $(x, y) \in L$.*

Lemma 2.6.2. *The foliation $\mathcal{F}_0^+$ in U_0^+ is a vertical foliation $x = const$ with leaves $x = p^{-k}(0)$ being double for all $k \geq 0$.*

Proof. For each point x_0 there exists n, such that in a neighborhood of the line $x = x_0$ the foliation $\mathcal{F}_0^+$ is determined by the level sets of the function $\varphi_{0,+}^{2^n}$.

Since $\varphi_{0,+}^{2^n}(x, y) = b_p^{2^n}(x)$, the foliation is vertical.

The multiple leaves appear when $\left(b_p^{2^n}\right)'(x) = 0$. $\left(b_p^{2^n}\right)'(x) = 0$ if and only if $p^n (x) = 0$. Moreover, all zeros of $\left(b_p^{2^n}\right)'$ are non-degenerate. Therefore, all the leaves $x = p^{-n}(0)$ are double. □

Lemma 2.6.3. *The foliation $\mathcal{F}_0^-$ in U_0^- consists of the leaves $\{p(y) - x = const\}$.*

Proof. By Lemma $\varphi_{0,-} = p(y) - x$ and is defined in U_0^-. The statement of the Lemma immediately follows. □

Corollary 2.6.1. $\mathcal{C}_0 = \left[\{y = 0\} \bigcup_k \{x = p^{-k}(0)\}\right] \cap U_0^+ \cap U_0^-$

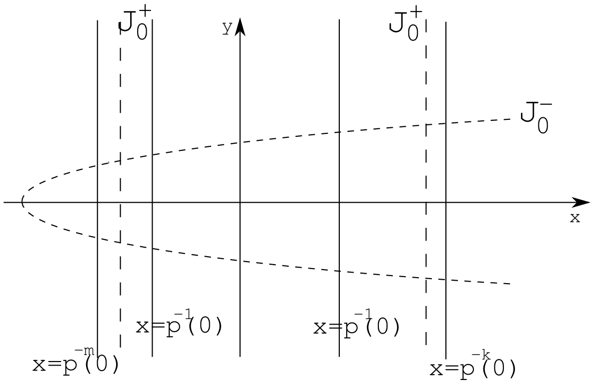


Figure 2.8: The degenerate Critical Locus

2.7 Critical locus near infinity.

The goal of this section is to calculate the critical locus at a neighborhood of $x = \infty$ in the compactification of $\mathbb{C}^2$ given by $\mathbb{CP}^1 \times \mathbb{CP}^1$.

First, we extend the foliation $\mathcal{F}_a^+$ to the line $x = \infty$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$ compactification of $\mathbb{C}^2$.

Let

$$\hat{V}_+ = \{(x,y) \in \mathbb{CP}^1 \times \mathbb{CP}^1 | \ (x,y) \in V_+ \text{ or } \{x = \infty, y \neq \infty\}\}$$

Let $t = \frac{1}{x}$, $v = \frac{1}{y}$.

Lemma 2.7.1. *The function* $\frac{\varphi_{a,+}}{x}$ *extends as a holomorphic function to* $\hat{V}_+ \times D_R$

Moreover, $t\varphi_{a,+} = 1 + th_1(t,y,a)$, $\frac{\partial h_1}{\partial y} = u\tilde{h}_1(t,y,a)$.

Proof. By the Riemann Extension Theorem, $\frac{\varphi_{a,+}}{x}$ can be extended to $x = \infty$.

Note that the functions s_n^+ and $\frac{\partial s_n^+}{\partial y}$ vanish in t for all k. Thus, in the sum (2.2) every term vanishes in u. Therefore, the infinite sum (2.2) vanishes in u as well. And $\frac{\varphi_{a,+}}{x} = 1 + th_1(t,y,a)$, where h_1 is a holomorphic function in t, y, a. The same way one proves, $\frac{\partial h_1}{\partial y} = t\tilde{h}_1(t,y,a)$. □

$$\hat{V}_- = \{(x,y) \in \mathbb{CP}^1 \times \mathbb{CP} | \ (x,y) \in V_- \text{ or } \{y = \infty, x \neq \infty\}\}$$

Lemma 2.7.2 ([LR])**.** *The function* $\frac{\varphi_{a,-}}{y}$ *extends holomorphically to* $\hat{V}_- \times D_R$*. Moreover,* $v\varphi_{a,-} = 1 + vh_2(x,v,a)$, $\frac{\partial h_2}{\partial x} = v\tilde{h}_2(x,v,a)$.

Proof. The proof is the same as in the previous lemma. □

In $V_+ \cap \mathcal{D}_{1,-}$ the critical locus is given by the zeroes of the form

$$\omega = d\varphi_{a,+} \wedge d\varphi_{a,-}^2 \wedge da$$

For the next lemma we fix (t,y)-coordinates in $\hat{V}_+$.

Lemma 2.7.3. *The critical locus extends holomorphically to* $\left(\hat{V}_+ \times D_R\right) \cap \mathcal{D}_{1,-}$*. The point* $(0, y, a) \in \mathcal{C}_a \cap \hat{V}_+ \cap \mathcal{D}_{1,-}$ *iff* $y = 0$*. The tangent line to the critical locus at* $(0, 0, a)$ *is given by* $2dy + Cdt = 0$ *with* C *depending on* a*.*

Proof.

$$d\varphi_{a,+} = -\frac{dt}{t^2} + \frac{\partial h_1}{\partial t}dt + \frac{\partial h_1}{\partial y}dy + \frac{\partial h_1}{\partial a}da = \frac{1}{t^2}\left(-1 + t^2\frac{\partial h_1}{\partial t}dt + t^3\tilde{h_1}dy + t^2\frac{\partial h_1}{\partial a}da\right)$$

$$\varphi^2_{a,-}(x, y) = a\varphi_{a,-} \circ f_a^{-1}(x, y) = a\varphi_{a,-}\left(y, \frac{p(y) - x}{a}\right) = \frac{tp(y) - 1}{t} + ah_2(y, \frac{at}{tp(y) - 1}, a)$$

$$d\varphi_{a,-} = \left(p'(y) + a\frac{\partial h_2}{\partial y}(y, \frac{at}{tp(y) - 1}, a) + \frac{a^2t^2p'(y)}{(tp(y) - 1)^2}\frac{\partial h_2}{\partial v}(y, \frac{at}{tp(y) - 1})\right)dy$$

$$+\left(\frac{1}{t^2} + \left[\frac{a}{tp(y) - 1} + \frac{atp(y)}{(tp(y) - 1)^2}\right]\frac{\partial h_2}{\partial v}\right)dt + h_3(t, y, a)da$$

,

where $h_3(t, y, a)$ is some holomorphic function in t, y, a.

By Lemma 2.7.2, $\frac{\partial h_2}{\partial x} = v\tilde{h}_2$. Hence $\frac{\partial h_2}{\partial x}(y, \frac{at}{tp(y)-1}, a) = \frac{at}{tp(y)-1}\frac{\partial h_2}{\partial x}(y, \frac{at}{tp(y)-1})$. Thus, there are holomorphic functions h_4, h_5 on $\mathcal{D}_1^-$ so that

$d\varphi_{a,-} = (p'(y) + th_4(t, y, a))dy + (\frac{1}{t^2}h_5(t, y, a))dt + h_3(t, y, a)da$

Therefore, there exists a holomorphic function h_6 on $\left(\hat{V}_+ \times D_R\right) \cap \mathcal{D}_{1,-}$ so that

$$\omega t^2 = (p'(y) + th_6)\, dy \wedge dt \wedge da$$

Conclusion follows. □

Corollary 2.7.1. *Fix* ε*. There exists* δ *so that for* $|a| < \delta$ *the critical locus in*

$$\{|y| \le \varepsilon\} \cap \{|x| \ge \alpha\}$$

is the graph of a function $y(x)$*.*

2.8 Horizontal and vertical invariant cones.

2.8.1 Horizontal cones.

Fix a domain B.

Definition 2.8.1. *A family of cones* C_x *in the tangent bundle to* B *is* f_a*-invariant iff for any point* $x \in B$*, such that* $f_a(x) \in B$*, we have* $df_a(C_x) \cup C_{f(x)}$*.*

Lemma 2.8.1. *Fix* $r' > r$ *and* β*. For every* $C < \min\{x : G_p(x) \le \frac{r'}{2}\}$ *there exists* δ *such that for all* $|a| < \delta$ *the family of horizontal cones* $|\xi| > C|\eta|$ *is* f_a*-invariant in* $\{G_a^+(x,y) \le r'\} \cap \{|y| \le \beta\}$ *(where* $(\xi, \eta) \in T_{(x,y)}\mathbb{C}^2$*).*

Note 2.8.1. *Note that we chose the box* $\{G_a^+(x,y) \le r'\} \cap \{|y| \le \beta\}$ *so that the tip of the parabola does not belong to the box. This allows us to have an invariant horizontal family of cones.*

Proof. $Df_a(x,y) = \begin{pmatrix} 2x & a \\ 1 & 0 \end{pmatrix}$

Let $\begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} = \begin{pmatrix} 2x & a \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$

We want to find C such that

$$|\xi| > C|\eta| \Rightarrow |2x\xi + a\eta| > C|\xi|.$$

Take $C = \min(2x) - \varepsilon$, where $b_p(x) \le \frac{r'}{2}$ and ε is any number. Then $\delta_2 = \varepsilon \max(2x)$.

□

2.8.2 Vertical cones.

Lemma 2.8.2. *Fix* $r' \ge r$*,* α*,* C*. There exists* δ *such that for all* $|a| < \delta$ *the family of cones* $|\xi| < C|a||\eta|$ *is* f_a^{-1}*-invariant in* $\{G_a^+(x,y) \le r'\} \cap \{|y| \le \alpha\}$*,* $(\xi, \eta) \in T_{(x,y)}\mathbb{C}^2$*.*

Proof. $Df_a^{-1} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{a} & \frac{2y}{a} \end{pmatrix}$

Let $\begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{a} & \frac{2y}{a} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$

$\left|\frac{\xi_1}{\eta_1}\right| = \frac{|a|}{|-\frac{\xi}{\eta}+2y|}$

Suppose $(x, y) = f_a^{-1}(u, v)$, where $\{(u, v)|\ G_a^+(u, v) \le r', |v| \le |\alpha|\}$, then $|y| > C_1$.

Therefore, when $|a| < \delta$, $|2y - \frac{\xi}{\eta}| \ge |C_1|$ ☐

2.9 Description of $\mathcal{F}_a^-$.

In this section we give a description of $\mathcal{F}_a^-$ in

$$W \cap \{|p(y) - x| \ge |a|\alpha\}.$$

We choose α in the definition of W such that the description of $\mathcal{F}_a^-$ is especially nice.

The function $\varphi_{a,-}^2$ is well-defined in this region, so it is natural to expect that the foliation

$$\mathcal{F}_a^- = \{\varphi_{a,-}^2 = \text{const}\}$$

is close to the foliation

$$\mathcal{F}_0^- = \{\varphi_{0,-}^2 = p(y) - x = \text{const}\}.$$

The only region where it really needs to be checked is when we approach $a|\alpha|$-neighborhood of C_p.

We also prove that the leaves of $\mathcal{F}_a^-$ that intersect the boundaries

$$\{G_a^+ \le r\} \cap \{|p(y) - x| = a|\alpha|\}$$

$$\{G_a^+ \leq r\} \cap \{|y| \leq |\alpha|\}$$

are horizontal-like. In order to guarantee this we will need to choose appropriate α.

We start by fixing preliminary $\tilde{\alpha}$ such that conditions (2.4.1) and (2.4.2) are satisfied.

Notice that $\varphi^2_{a,-}$ is well-defined in

$$f_a(V_-) = \{|p(y) - x| \geq |a|\tilde{\alpha}, |p(y) - x| \geq |a||y|\}$$

.

Therefore, the domain of definition of $\varphi^2_{a,-}$ in W is

$$f_a(V_-) \cap W = \{|p(y) - x| \geq |a|\tilde{\alpha}\}.$$

Lemma 2.9.1. *There exists α such for all $a \in D_R$*

$$\min\{|\varphi^2_{a,-}(x,y)| : \ (x,y) \in W, |p(y) - x| = \alpha|a|\} >$$

$$\max\{|\varphi^2_{a,-}(x,y)| : \ (x,y) \in W, |p(y) - x| = \tilde{\alpha}|a|\}.$$

Proof. $\varphi^2_{a,-}(x,y) = a\varphi_{a,-}(y, \frac{p(y)-x}{a})$.

$\varphi_{a,-}(x,y) \sim y$ as $y \to \infty$, Therefore,

$$\min\{|\varphi_{a,-}(x,y)| : \ |x| < \theta, |y| = \alpha\} > \max\{|\varphi_{a,-}(x,y)| : \ |x| < \theta, |y| = \tilde{\alpha}\}$$

for big enough α. Take $\theta = \max(p^{-1}(D_{|x|+|a|\alpha}))$.

□

Corollary 2.9.1. *For a point $q \in W \cap \{|p(y) - x| \geq |a|\alpha\}$ a connected component of a leaf L_q of the foliation $\mathcal{F}^-_a$, passing through a point q, stays outside of $\tilde{\alpha}|a|$-neighborhood of C_p.*

Lemma 2.9.2. *For all* $a \in D_R$, $\frac{\partial\varphi^2_{a,-}/\partial y}{\partial\varphi^2_{a,-}/\partial x}$ *is* a*-close to* $\frac{\partial\varphi^2_{0,-}/\partial y}{\partial\varphi^2_{0,-}/\partial x} = -p'(y)$ *in* Ω.

Note 2.9.1. *Note that* $\frac{\partial\varphi^2_{a,-}}{\partial y}, \frac{\partial\varphi^2_{a,-}}{\partial y}$ *do not stay bounded in* Ω *as* $a \to 0$, *but their ratio does.*

Proof. We chose α so that the function $\varphi^2_{a,-}$ is well-defined in $\{G^+_a \le r\} \cap \{|a|\tilde\alpha \le |p(y)-x| \le \kappa\}$ and that the leaves of $\mathcal{F}^-_a$ we consider do not leave this neighborhood.

Let us show that in this region $\frac{\partial\varphi^2_{a,-}/\partial y}{\partial\varphi^2_{a,-}/\partial x}$ is a-close to $\frac{\partial\varphi^2_{0,-}/\partial y}{\partial\varphi^2_{0,-}/\partial x}$.

$$\frac{\partial\varphi^2_{a,-}}{\partial y}(x,y) = \frac{\partial}{\partial y}\left(a\varphi_{a,-}(y, \frac{p(y)-x}{a})\right) = a\frac{\partial\varphi_{a,-}}{\partial x}\left(y, \frac{p(y)-x}{a}\right) +$$

$$p'(y)\frac{\partial\varphi_{a,-}}{\partial y}\left(y, \frac{p(y)-x}{a}\right)$$

$$\frac{\partial\varphi^2_{a,-}}{\partial x}(x,y) = \frac{\partial}{\partial x}\left(a\varphi_{a,-}(y, \frac{p(y)-x}{a})\right) = -\frac{\partial\varphi_{a,-}}{\partial y}$$

Let us do a change of coordinates $(u,y,v) = (p(y)-x, y, \frac{a}{p(y)-x})$. First, we introduce the u-coordinate that measures the distance to parabola. Then we do a blow-up in each line $y = \mathit{const}$ that blows-up a cone in u, a-coordinates, which corresponds to the compliment of $|a|\alpha$ neighborhood, to a polydisk in u, v coordinates.

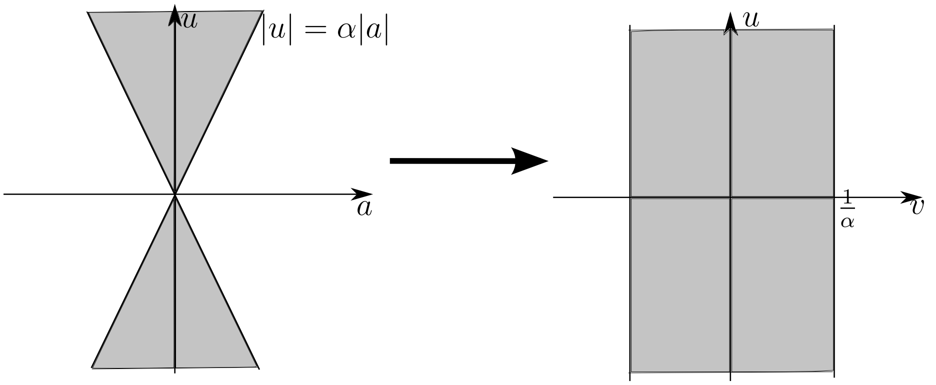


Figure 2.9: The blow-up

Denote by $\tilde\varphi^2_-(u,y,v) = \varphi^2_{uv,-}(p(y)-u,y)$. We show that a function $\frac{\partial\tilde\varphi^2_-/\partial y}{\partial\tilde\varphi^2_-/\partial x}$ lifts to a holomorphic function on the blown-up space $\{|y| < \alpha, |u| < \varepsilon, |v| < \frac{1}{\alpha}\}$.

$$a\frac{\partial\varphi_{a,-}}{\partial x}(y,\frac{p(y)-x}{a}) = uv\frac{\partial\varphi_{uv,-}}{\partial x}(y,\frac{1}{v})$$

$$\lim_{u\to 0} uv\frac{\partial\varphi_{uv,-}}{\partial x}(y,\frac{1}{v}) = 0$$

$$\lim_{u\to 0}\frac{\partial\tilde{\varphi}^2_-/\partial y}{\partial\tilde{\varphi}^2_-/\partial x} = -p'(y).$$

Thus, $\frac{\partial\varphi^2_{a,-}/\partial y}{\partial\varphi^2_{a,-}/\partial x}$ is a-close to $\frac{\partial\varphi^2_{0,-}/\partial y}{\partial\varphi^2_{0,-}/\partial x}$ in $\Omega\cap\{|p(y)-x|\geq|a|\alpha\}$. □

Let L_q be a leaf of foliation $\mathcal{F}^-_a$ that passes through a point q.

Corollary 2.9.2. *There exist κ and δ such that for all $|a|<\delta$ and all*

$$q\in\{G^+_a\leq r\}\cap\{|p(y)-x|\geq|a|\alpha\}\cap\{|y|\geq\kappa\}$$

a connected component of L_q is horizontal-like.

Proof. One takes κ such that every leaf of $\mathcal{F}^-_0$ that intersects $|y|=\kappa$ is horizontal like. □

Corollary 2.9.3. *There exists δ so that for all $|a|<\delta$ and all $(x,y)\in\Omega$ the tangent plane to the foliation $\mathcal{F}^-_a$ is not horizontal.*

2.10 Description of $\mathcal{F}^+_a$.

Definition 2.10.1. *We say that a curve C in a domain B is horizontal-like iff there exists a family of f_a-invariant horizontal cones in B, such that the tangent lines to C belong to this family.*

The function $G^+_0(x,y)=G_p(x)$ is self-similar:

$$G_p(p(x)) = 2G_p(x).$$

Recall that we chose r so that

$$G_p(0) < r < G_p(p(0)).$$

The picture of the level sets of G_p inside $\{G_p < r\}$ is self-similar. Inside each connected component of $\{G_p = \frac{r}{2^n}\}$ there are exactly two connected components $\{G_p = \frac{r}{2^{n+1}}\}$.

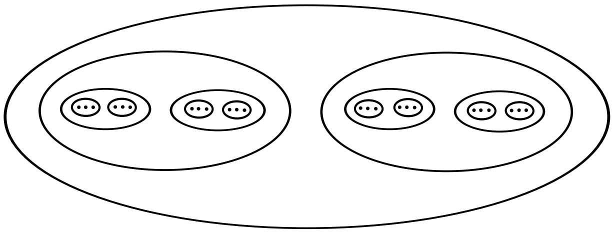


Figure 2.10: Level sets $G_p = \frac{r}{2^n}$.

There is exactly one critical level in each connected component

$$\frac{r}{2^{n+1}} \leq G_p \leq \frac{r}{2^n}.$$

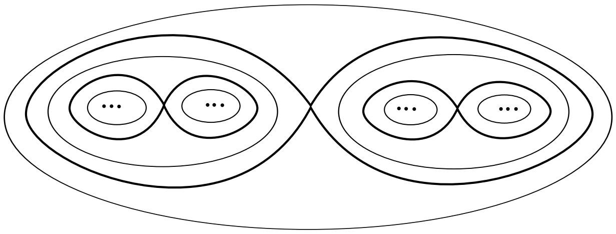


Figure 2.11: Level sets of G_p.

In this section we show that the picture persists for the level sets of G_a^+ for small enough a on each horizontal-like curve in a box

$$\{G_a^+ \leq r\} \cap \{|y| \leq \beta\}.$$

We choose $\beta = 2\max\{p(x)| \quad G_p(x) \leq r\}$ to apply this construction to the leaves of $\mathcal{F}_a^-$ that intersect $\{|y| = \alpha\}$ in $\{G_a^+ \leq r\}$.

Lemma 2.10.1. *There exists δ such that for $|a| < \delta$ $\{G_a^+ = r\}$, $\{G_a^+ = \frac{r}{2}\}$ are non-critical on each horizontal-like curve inside $|y| < \beta$. Moreover, $\{G_a^+ = r\}$ has one connected component, $\{G_a^+ = \frac{r}{2}\}$ has two connected components.*

Proof. G_a^+ is a function that depends analytically on a on $\mathcal{U}_+$ for all $a \in D$. Therefore, since the level sets $\{G_0^+ = r\}$ and $\{G_0^+ = \frac{r}{2}\}$ on $y = b$ are non-degenerate, they remain non-degenerate for small enough a. The same is true for the number of components. □

Lemma 2.10.2. *There exists δ so that for all $|a| < \delta$, there is exactly one critical level of G_a^+ between $\frac{r}{2}$ and r on each horizontal-like curve in $|y| < \beta$. The corresponding critical point is non-degenerate.*

Proof. Let C be a horizontal-like curve. Then the domain $\{G_a^+ \leq r\}$ inside C is parametrized by a planar domain. Therefore, the index of gradG_0^+ along the boundary of $\{\frac{r}{2} \leq G_0^+ \leq r\}$ is well-defined and is equal to one. The function G_a^+ depends holomorphically on a. Thus, the index of gradG_a^+ along the boundary of $\{\frac{r}{2} \leq G_a^+ \leq r\}$ is one as well for small a. Therefore, there is only one critical point inside and it is non-degenerate. □

Lemma 2.10.3. *There exists δ so that for all $|a| < \delta$ $T_r = \{G_a^+ = r\} \cap W$ is a solid torus, $\{G_a^+ = \frac{r}{2}\} \cap W$ consists of two solid tori $T_{r/2}^1, T_{r/2}^2$ (the core coordinate can be chosen to be real-analytic, the disk coordinate holomorphic).*

Proof. $\{G_a^+ = r\} = \{(\varphi_{a,+}, y), |\varphi_{a,+}| = r, |y| \leq \alpha\}$.

For $\{G_a^+ = \frac{r}{2}\} \cap W$ the proof goes the same way.

□

Take any horizontal-like curve. We want to prove by induction that inside each component $\{G_a^+ = \frac{r}{2^n}\}$ there are exactly two components $\{G_a^+ = \frac{r}{2^{n+1}}\}$, and they are non-critical. Therefore, there is exactly one critical level in between, and the corresponding critical point is non-degenerate.

Lemma 2.10.4. *There exists δ so that for all $|a| < \delta$, the level set $\{G_a^+ = \frac{r}{2^n}\}$ on each horizontal-like curve in $|y| < \beta$ is non-critical.*

Proof. $f_a^n(\{G_a^+ \leq \frac{r}{2^n}\})$ is horizontal-like, since it is an image of a horizontal-like curve and it belongs to the box with f_a-invariant horizontal cones.

$f_a^n(G_a^+ = \frac{r}{2^n}) \in T_r$ and it projects one-to-one to x-axis. Therefore, it is non-critical.

□

Lemma 2.10.5. *For $|a| < \delta$ on each horizontal-like curve for every n there are exactly two level sets $\{G_a^+ = \frac{r}{2^{n+1}}\}$ inside $\{G_a^+ = \frac{r}{2^n}\}$ and they are non-critical.*

Proof. For every n, $f_a^n(\{G_a^+ \leq \frac{r}{2^n}\})$ is a disk that projects one-to-one to x-axis with the boundary on T_r. It intersects $T^1_{r/2}$, $T^2_{r/2}$ by a circle each. On the intersection $\{G_a^+ = \frac{r}{2}\}$. This proves the lemma. □

Corollary 2.10.1. *For $|a| < \delta$ on each horizontal-like curve there is only one critical level $\{G_a^+ = r'\}$, where $\frac{r}{2^{n+1}} < r' < \frac{r}{2^n}$ for each connected component $\{G_a^+ = \frac{r}{2^n}\}$.*

The next lemma states that the foliation $\mathcal{F}_a^+$ is not only vertical-like (projects one-to-one to y-axis), but is uniformly close to vertical on some thickening of $\{G_a^+ = \frac{r}{2^n}\}$.

Lemma 2.10.6. *Fix a small λ. There exists δ s.t. $\forall |a| < \delta$*

$$\left| \frac{\partial \varphi_{a,+}^{2^n}/\partial y}{\partial \varphi_{a,+}^{2^n}/\partial x} \right| < C|a|$$

on $\{\frac{r-\lambda}{2^n} \leq G_a^+ \leq \frac{r+\lambda}{2^n}\}$ with C independent on n.

Proof. On $\{r - \lambda \leq G_a^+ \leq r + \lambda\}$ the inequality follows from the fact that $\varphi_{a,+}$ is a holomorphic function in a.

The leaves of foliation $\mathcal{F}_a^+$ in $\{\frac{r-\lambda}{2^{n+1}} \leq G_a^+ \leq \frac{r+\lambda}{2^{n+1}}\} \cap W$ are preimages under f_a^{-1} of the leaves of $\mathcal{F}_a^+$ in $\{\frac{r-\lambda}{2^n} \leq G_a^+ \leq \frac{r+\lambda}{2^n}\} \cap W$. Therefore, by induction we check that they belong to f_a^{-1}-invariant vertical cones. □

2.11 Critical Locus in Ω.

Recall that

$$\Omega = \{G_a^+ \leq r\} \cap \{|y| \leq \alpha\} \cap \{|p(y) - x| \geq |a|\alpha\}.$$

See Figure 2.3.

In this section we subdivide Ω into countably many regions. We call Ω_{ξ_n} $(p^n(\xi_n) = 0)$ the region that is a connected component of

$$\{\frac{r}{2^{n+1}} \leq G_a^+ \leq \frac{r}{2^n}\} \cap \{|y| \leq \alpha\} \cap \{|p(y) - x| \geq |a|\alpha\}.$$

that contains a line $x = \xi_n$ when $a = 0$.

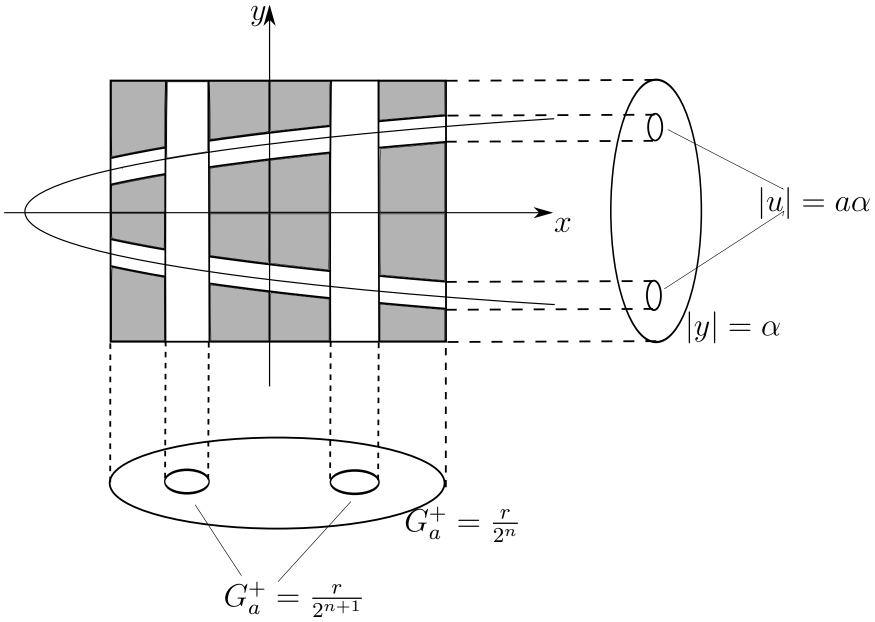


Figure 2.12: Domain Ω_{ξ_n}

Lemma 2.11.1. *Fix a small λ. There exists δ (independent on n) so that for all $|a| < \delta$ the critical locus in each connected component of*

$$\{\frac{r - \lambda}{2^n} \leq G_a^+ \leq \frac{r + \lambda}{2^n}\} \cap \{|y| \leq \alpha\}$$

is an annulus which is a graph of function $y(\varphi_{a,+})$.

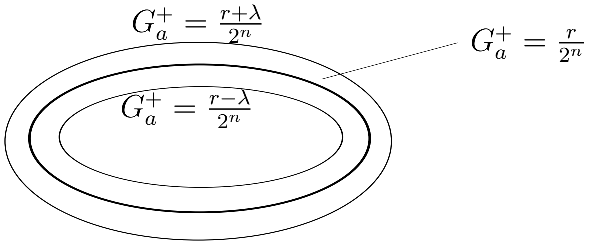


Figure 2.13: The thickening of $G_a^+ = \frac{r}{2^n}$

Proof. Fix γ so that the set $\{G_0^+ \le r\} \cap \{|y| \le \gamma\}$ is disjoint from C_p. Then for all $|a| < \delta'$ the set $\{G_a^+ \le r\} \cap \{|y| \le \gamma\}$ is disjoint from C_p.

The inverse function theorem implies that $\forall n\ \exists \delta_n$ s.t. $|a| < \delta_n$ the critical locus in $\{\frac{r-\lambda}{2^n} \le G_a^+ \le \frac{r+\lambda}{2^n}\} \cap \{|y| \le \gamma\}$ is an annulus on the component that is a perturbation of $y = 0$.

Since in the region

$$\{\frac{r-\lambda}{2^n} \le G_a^+ \le \frac{r+\lambda}{2^n}\} \cap \{\gamma < |y| < \alpha\} \cap \{|p(y) - x| \ge a|\alpha|\}$$

the foliation $\mathcal{F}_a^-$ is a-close to $\mathcal{F}_0^-$ and by Lemma 2.10.6 $\mathcal{F}_a^+$ is almost vertical, for $|a| < \delta''$ there are no points of the critical locus in this region.

Take some n and fix some connected component

$$\{\frac{r-\lambda}{2^n} \le G_a^+ \le \frac{r+\lambda}{2^n}\} \cap \{|y| \le \gamma\}.$$

Let us show that the critical annuli in this connected component persists for all $|a| \le \delta''$. $\mathcal{C}_a$ is a analytic set in the region

$$\{(x, y, a)|\ \frac{r-\lambda}{2^n} \le G_a^+(x, y) \le \frac{r+\lambda}{2^n}, |y| < \gamma, |a| < \delta''\}$$

There are no zeroes of gradG_a^- on the leaves of $\mathcal{F}_a^+$ on curve $|y| = \gamma$. Therefore, index of gradG_a^- is constant. At $a = 0$ it is equal to 1. Thus, the critical annulus in

$$\{(x, y, a)|\ \frac{r-\lambda}{2^n} \le G_a^+(x, y) \le \frac{r+\lambda}{2^n}, |y| < \gamma, |a| < \delta''\}$$

persists. ☐

Lemma 2.11.2. *There exist κ and δ such that for all $|a| < \delta$ the critical locus in each connected component $\{|a|\alpha \leq |p(y) - x|\} \cap \{\frac{r}{2^{n+1}} \leq G_a^+ \leq \frac{r}{2^n}\} \cap \{|y| \geq \kappa\}$ is an annulus with two holes and is a graph of function $y(\varphi_{a,-})$.*

Proof. It follows from Corollary 2.9.2 ☐

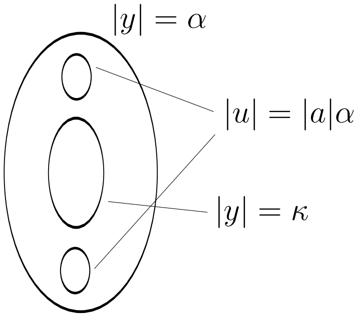


Figure 2.14: The critical locus in $\{|a|\alpha \leq |p(y) - x|\} \cap \{\frac{r}{2^{n+1}} \leq G_a^+ \leq \frac{r}{2^n}\} \cap \{|y| \geq \kappa\}$

Lemma 2.11.3. *For $|a| < \delta$ the critical locus in Ω is a smooth curve. In Ω_{ξ_n} it is a connected sum of two disks D_1, D_2 with two holes. The boundary of D_1 belongs to $\{|y| = \alpha\}$, and the holes of D_1 have boundaries on $\{|u| = |a|\alpha\}$. The boundary of D_2 belongs to $\{G_a^+ = \frac{r}{2^n}\}$ and the holes to $\{G_a^+ = \frac{r}{2^{n+1}}\}$.*

Note 2.11.1. *As $a \to 0$, the curve degenerates to $(x - \xi_n)y = 0$. The holes of D_1 degenerate to points $(0, \xi_{n+1})$, $(0, \xi'_{n+1})$, where $p(\xi'_{n+1}) = p(\xi'_{n+1}) = \xi_n$. The holes of D_2 tend to circles $\{(x, 0)|\ G_p(x) = \frac{r}{2^{n+1}}\}$.*

Proof. Fix n. The union of two intersecting lines $y = 0$ and $x = \xi_n$ has Milnor number 1 [AGV85]. Therefore, in the neighborhood of a point $(\xi_n, 0)$ the critical locus for small a is either a handle, or union of two intersecting curves. By the inverse function theorem and Lemma 2.11.2 there exists δ_n s.t. $|a| < \delta_n$ this critical locus can be extended all the way to the boundary. Let us show that it can not be union of two intersecting curves.

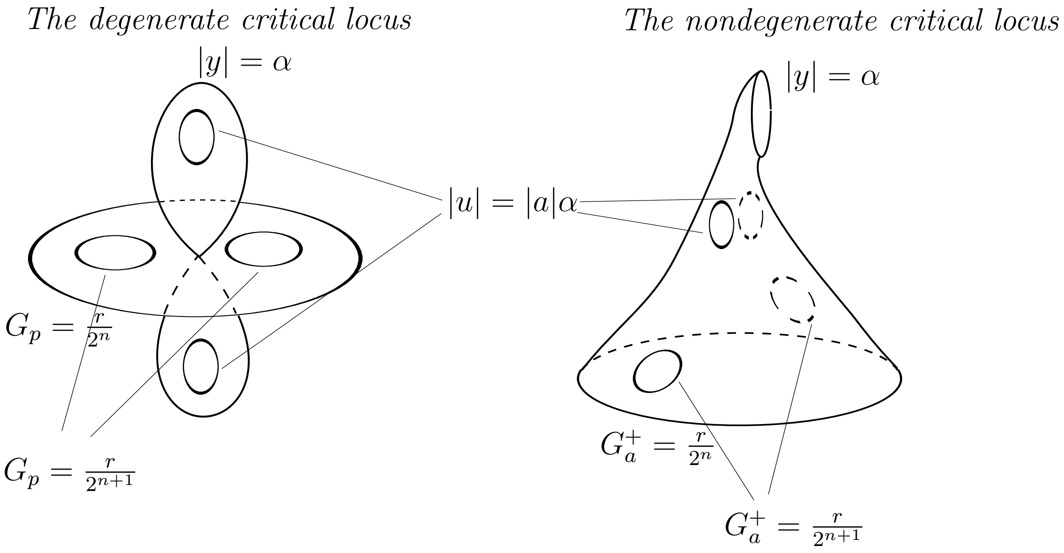


Figure 2.15: The critical locus in Ω_{ξ_n}

Take a 'horizontal' curve, the curve that goes from one vertical boundary to the other. This curve necessarily intersects the 'vertical' curve on which the tangent line to $\mathcal{F}_a^+$ are horizontal. But then on the point of intersection the tangent line to the foliation $\mathcal{F}_a^-$ is horizontal. That is impossible by Lemma 2.9.3.

Let us show that the picture persists for $|a| < \delta$. We have trapped critical locus on the boundary in Lemmas 2.11.1, 2.11.2. We have shown as well that no new component can enter through the boundary. The component inside can degenerate only to union of two intersecting curves which is impossible by the same argument as before. □

2.12 Extension of the critical locus up to $|a|\alpha$-neighborhood of parabola.

Lemma 2.12.1. *There exists δ so that for all $|a| < \delta$ the component of the critical locus, that is a perturbation of $y = 0$ can be extended up to $|a|\alpha$-neighborhood of parabola as a graph of function $y(x)$.*

Proof. The domain of definition of $\varphi^2_{a,-}$ is $f_a(V_-) = \{(x, y) \mid |p(y) - x| \geq |a|\alpha, |p(y) - x| \geq$

$|a||y|\}$.

Therefore, in W the domain of definition of $\varphi^2_{a,-}$ is $\{|p(y)-x| \geq |a|\alpha\}$.

Denote $u = p(y) - x$.

Consider new variables $(u, y, v) = (u, y, \frac{a}{u})$. Denote by π projection

$\pi : (u, y, v) \to (u, y, uv)$.

Notice that π^{-1} blows up a point $u = 0$ on each line $y = y_0$.

Let's prove that one can extend the critical locus to

$$S = \{|u| < \beta, |y| < \varepsilon, |v| < \frac{1}{\alpha}\}.$$

Note that $\varphi_{a,+}$ can be extended to S since it's a well-defined holomorphic function in $\pi(S)$. $\varphi^2_{a,-}(x, y) = uv\varphi_{uv,-}(y, 1/v)$.

By [LR] $\frac{\varphi_{a,-}(x,y)}{y}$ extends to be a holomorphic function to a neighborhood of $y = \infty$.

Therefore $\varphi^2_{a,-}$ extends to S. Moreover, notice that on blown-up lines $\varphi^2_{uv,-} = 0$.

The critical locus is given by the zeroes of the function

$$w = \frac{d\varphi_{a,+} \wedge d\varphi^2_{a,-} \wedge da}{dx \wedge dy \wedge da}.$$

Denote by

$$\tilde{w} = -\frac{d\varphi_{uv,+} \wedge d\varphi^2_{uv,-} \wedge dv}{udu \wedge dy \wedge dv}.$$

Notice that $\tilde{w} = w \circ \pi$.

$$\tilde{w} = -uv\frac{\partial\varphi_{uv,+}}{\partial x}(p(y) - u, y)\frac{\partial\varphi_{uv,-}}{\partial x}(y, \frac{1}{v}) +$$

$$\left(\frac{\partial\varphi_{uv,+}}{\partial x}(p(y) - u, y)p'(y) + \frac{\partial\varphi_{uv,+}}{\partial y}(p(y) - u, y)\right)\frac{\partial\varphi_{uv,-}}{\partial y}(y, \frac{1}{v})$$

Note that $\frac{\varphi_{a,-}}{y} = 1 + aH(x, \frac{1}{y}, a)$, where H is a holomorphic function in some neighborhood of $(x, 0, 0)$.

$\lim_{u\to 0} v\frac{\partial \varphi_{uv,-}}{\partial x} = 0$

$\lim_{u\to 0} \frac{\partial \varphi_{uv,-}}{\partial y} = 1$

$$\tilde{w}(0,y,v) = b'_p(p(y))p'(y)$$

Note that for all $|v| < \frac{1}{\alpha}$ $|y| < \alpha$ $\tilde{w}(0,y,v) = 0$ only when $y = 0$ and the zero is not multiple. Therefore, by Weierstrass theorem for v $y = g(u,v)$.

□

2.13 Description of the critical locus.

Fix some ε. Denote

$$\hat{\Omega}_1 = \{(x,y) \in \mathbb{C}P^2 |\ |y| < \varepsilon, |p(y) - x| \geq |a|\alpha, G^+_a(x,y) \geq r\}.$$

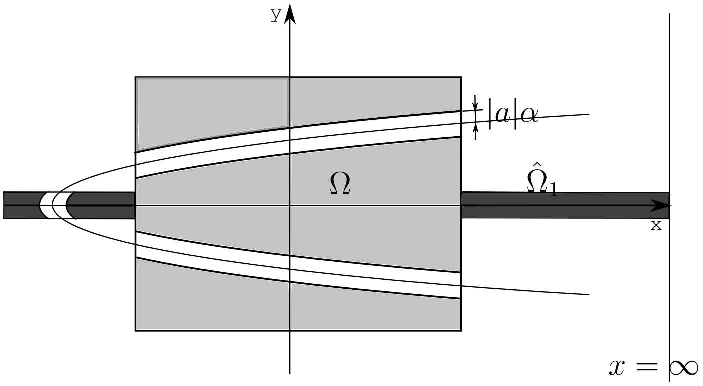


Figure 2.16: Domain $\hat{\Omega}_1$

Lemma 2.13.1. *There exists* δ *such that* $\forall |a| < \delta$ *the critical locus* $\mathcal{C}_a$ *in* $\hat{\Omega}_1$ *is a punctured disk, with a hole removed. The puncture is at the point* $(\infty, 0)$*, the boundary of the hole belongs to* $\{|p(y) - x| = |a|\alpha\}$*.*

Proof. By Lemma 2.6.1 the degenerate critical locus $\mathcal{C}_0$ in $\hat{\Omega}_1$ is $y = 0$, with point $x = c$ removed. By Lemma 2.7.1 it persists in some neighborhood of $x = \infty$. By the inverse function theorem it can be extended as a graph of function $y(x)$ to $\{ G_a^+(x,y) \geq r \}$ excluding ε-neighborhood of C_p. By Lemma 2.12.1 it can be extended up to $a|\alpha|$-neighborhood of C_p. ☐

In the following 3 lemmas we show that every component of the critical locus intersects Ω. It follows from Lemma 2.11.3 that it consists of one component.

Lemma 2.13.2. *Let C_a be a component of the critical locus $\mathcal{C}_a$. Then there exists a point on ∂C_a that belongs to $J_a^+ \cup J_a^-$.*

Proof. Consider $(G_a^+ + G_a^-)$. This function is pluriharmonic and strictly positive in $U_a^+ \cap U_a^-$. Therefore, $\inf (G_a^+ + G_a^-)$ cannot be attained at the interior point. ☐

Lemma 2.13.3. *There exists δ such that for all $|a| < \delta$ $J_a^+ \cap \Omega$ is a fundamental domain for $J_a^+ \backslash J_a$.*

Proof. [HOV95] There it is prove for Hénon mappings that are perturbations of hyperbolic polynomials with connected Julia set. In the connected case the proof is the same. ☐

Lemma 2.13.4. *Suppose $|a| < \delta$ and C_a is a component of the critical locus. Then there exists an iterate of $f_a^n(C_a)$ that intersects Ω.*

Proof. Lemma 2.13.2 states that there exists $z \in (J_a^+ \cup J_a^-) \cap (\partial C_a)$. Suppose $z \in J_a^-$. Take a sequence of points $z_n \in C_a$, $z_n \to z$.

$G_a^-(z_n) \to 0$ as $n \to \infty$; G_a^+ is bounded.

For every n there exists k_n such that $1 < G_a^-(f_a^{-k_n}(z_n)) \leq 2$. Then $k_n \to \infty$.

$$G_a^+(f_a^{-k_n}(z_n)) \to 0.$$

Taking a subsequence one may assume $f^{-k_n}(z_n) \to z$. $z \in J_a^+ \backslash J_a$. Since by Lemma 2.13.3 $f_a^m(z) \in J_a^+ \cap \Omega$, there exists an iterate of C_a that intersects Ω. ☐

Below we will prove that $\mathcal{C}_a \cap \left(\Omega \cup \hat{\Omega}_1\right)$ is a fundamental domain of the critical locus.

The next lemma is a variation of the Classical Poincaré's Polyhedron Theorem on the fundamental domain under the group action ([M88], IV.H).

Lemma 2.13.5. *Let C be a Riemann surface. Let $f : C \to C$ be an automorphism. Let $D \subset C$ be an open domain. Suppose*

1. $f_a^n(D) \cap D = \emptyset$;
2. $\partial D \subset C$ *consists of countably many smooth curves* γ_i;
3. γ_i *are being paired by the map* f;
4. *for any sequence* $\{z_n\}$, $z_n \in D$, $f_a^n(z_n)$ *does not have an accumulation point in* S

Then D is a fundamental domain of S for the map f_a.

Proof. Denote by S a Riemann surface obtained by gluing the images $f^n(D)$ to D. The natural map $i : S \to C$ is injective by (1). Let us prove that it is proper. If it is not, then there exists a sequence of points $z_n \in S$ so that $z_n \to \partial S$, $i(z_n) \to z \in C$. Take $z_n' \in D$ so that $z_n = f^{k_n}(z_n')$. If there are infinitely many same k_n's. Then the sequence $\{z_n'\}$ has an accumulation point in D. That contradicts (2) and (3).

Therefore, taking a subsequence one can assume $\{k_n\}$ increase. This contradicts (4). □

Lemma 2.13.6. *The images of $(W \backslash \{|p(y) - x| \geq a|\alpha|\}) \cup \{|y| < \varepsilon\}$ under the map f_a are disjoint.*

Proof. 1.

$$f_a^n(W \backslash \{|p(y) - x| \leq |a|\alpha\}) \cap [W \backslash \{|p(y) - x| \leq a|\alpha|\}] = \emptyset.$$

$f_a(W) \subset W \cup V_+$, $f_a(V_+) \subset V_+$, therefore, $(f_a^n(W) \cap W) \subset (f_a(W) \cap W) \subset \{|p(y) - x| \leq |a|\alpha\}$

2.

$$f_a^n(W) \cap \{|y| < \varepsilon\} \cap \{|x| \geq \alpha\} = \emptyset;$$

Let $(x, y) \in W$. Suppose $f_a^n(x, y) \in V_+$ and n is the smallest number such that this happens. Then $|p(x_n) - y_n| \leq |a|\alpha$. Thus, $|y_n| > \varepsilon$. For points in V_+, $|y_{n+1}| > |y_n|$.

3.

$$f_a^n(\{|y| < \varepsilon\} \cap \{|x| > \alpha\}) \cap \{|y| < \varepsilon\} \cap \{|x| > \alpha\} = \emptyset.$$

Suppose $(x, y) \in \{|y| < \varepsilon\} \cap \{|x| > \alpha\}$. Since $|y| < \varepsilon$, $f_a(x, y)$ belongs to $|a|\varepsilon$-neighborhood of parabola. Since $(x, y) \in V_+$, $f_a(x, y) \in V_+$. Therefore, $|y_1| > \varepsilon$. $|y_{n+1}| > |y_n| > \varepsilon$, since $(x_n, y_n) \in V_+$.

4.

$$f_a^n(\{|y| < \varepsilon\} \cap \{|x| > \alpha\}) \cap W = \emptyset.$$

This is true, since $\{|y| < \varepsilon\} \cap \{|x| > \alpha\} \subset V_+$, and $f_a(V_+) \subset V_+$.

□

Lemma 2.13.7. *There exists δ such that for all $|a| < \delta$ the critical locus in $\Omega \cup \hat{\Omega}_1$ forms a fundamental domain of $\mathcal{C}_a$.*

Proof. Denote by $D = \mathcal{C}_a \cap \left(\Omega \cup \hat{\Omega}_1\right)$. Let us check that the conditions of the Lemma 2.13.5 are satisfied. Since $\Omega \cup \hat{\Omega} \subset (W \backslash \{|p(y) - x| \leq |a|\alpha\}) \cup \{|y| \leq \varepsilon\}$. All the forward images of D are disjoint from it. Therefore, (1) is satisfied.

The boundary of D in each Ω_{ξ_k} consists of vertical circles: $|y| = \alpha$ and $|p(y) - x| = |a|\alpha$. The circles $|p(y) - x| = |a|\alpha$ are parametrized by ξ'_{k+1}, ξ''_{k+1},$(p(\xi'_{k+1}) = p(\xi''_{k+1}) = \xi_k)$, which stands for the approximate value of y on the circle.

There is also one horizontal circle $|p(y) - x| = |a|\alpha$ on a perturbation of $y = 0$.

f_a maps $|y| = \alpha$ in Ω_{ξ_k} to $|p(y) - x| = |a|\alpha$ in $\Omega_{p(\xi_k)}$, parametrized by ξ_k

f_a maps $|y| = \alpha$ on perturbation of $x = 0$ to a horizontal circle $|p(y) - x| = |a|\alpha$.

Therefore, boundary components of D are being paired by f_a. And condition (2) and (3) of Lemma 2.13.5 are satisfied.

Suppose there exists a sequence of $z_n \in D$ so that $f_a^n(z_n) \to \partial S$, $z_n \to z_* \in C$.

If $\{z_n\}$ has an accumulation point z in D. Then $f_a^n(z) \to z^*$. Which is impossible, since $f_a^n(z) \to \infty$ in $\mathbb{C}^2$.

If $\{z_n\}$ does not have an accumulation point z in D. Then z_n accumulate to $z \in J_a^+$. Therefore, $f_a^{k_n}(z') \to z \in J_a$. That contradicts to $z \in C$.

So condition (4) is satisfied as well. Therefore, D is a fundamental domain of the critical locus $\mathcal{C}_a$. □

Proof of theorem 2.2.1. To obtain a description in terms of truncated spheres we do a dynamical regluing. We fix some small ε and cut the fundamental domain of $\mathcal{C}_a$ along the hypersurface $|y| = \varepsilon$. We call the connected component of the perturbation of $y = 0$ main component. The rest of components we call handles: H_{ξ_k} is a component in Ω_{ξ_k}

The boundary of H_{ξ_k} consists of four circles:

$|y| = \alpha$, $|y| = \varepsilon$ and two connected components of $|p(y) - x| = |a|\alpha$, parametrized as previously by ξ'_{k+1}, ξ''_{k+1}, where $p(\xi'_{k+1}) = p(\xi''_{k+1}) = \xi_k$.

We glue H_{ξ_k} to the main component by the map f_a^k. Under this procedure the boundary $|y| = \alpha$ of H_{ξ_k} is being glued to $|p(y) - x| = |a|\alpha$-boundary of $H_{p(\xi_k)}$, parametrized by ξ_k. By "generalized uniformization theorem", it can be straighten to be a sphere.

The fundamental domain of the critical locus $\mathcal{C}_a$, obtained after regluing, is a truncated sphere.

The preimages of 0 under the map p^k are parametrized by k-strings of 0 and 1's.

Let α_k be a k-string that parametrizes ξ_k. V_{α_k} is the interior of $|y| = \varepsilon$ obtained by cutting H_{ξ_k} from the main component. U_{α_k} is the interior of $f_a^k(|y| = \varepsilon)$ on $f_a^k(H_{\xi_k})$. The rest of the boundary corresponds to $G_a^+ = 0$ and $G_a^- = 0$. Therefore, it is parametrized by two Cantor sets Σ, Ω.

We show these are true Cantor sets by moduli counting.

Lemma 2.13.8. *Let* $U_1 \supset U_2 \supset \dots U_n \supset \dots$ *be a sequence of open domains, such that* $U_i \backslash U_{i-1}$ *is an annulus with moduli* $M_i \geq M$*. Then* $\bigcap \bar{U}_i$ *is a point.*

Take a point $\sigma \in \Sigma$. Let M_1 be a modulus of the annulus $\{r - \lambda \leq G_a^+ \leq r\}$ on $y = 0$. Then M_1 is a modulus of the annulus $\{\frac{r-\lambda}{2^n} \leq G_a^+ \leq \frac{r}{2^n}\}$. The main component project one-to-one to each of this annulus. There is a sequence of connected component $\{\frac{r-\lambda}{2^n} \leq G_a^+ \leq \frac{r}{2^n}\}$ that bound a hole parametrized by σ. Therefore, σ is a point. Σ is a Cantor set.

Fix some small ε. Let M_1 be a modulus of the annulus $\{\alpha - \varepsilon \leq |y| \leq \alpha\}$. All handles H_{ξ_k} project one-to-one to this annulus. So by the same argument we get that Ω is a Cantor set.

□

2.14 List of standard notations.

We provide a list of notations here as a reference

Notation	**Section**	**Meaning**
f_a	2.1	Hénon mapping under consideration
a	2.1	Jacobian of Hénon mapping under consideration
$p(x)$	2.1	polynomial used in the definition of Hénon mapping
U_a^+, U_a^-	2.1	set of points whose orbits under forward (backward) iteration of f_a escape to infinity
K_a^+, K_a^-	2.1	set of points whose orbits under forward (backward) iteration of f_a remain bounded
J_a^+, J_a^-	2.1	boundaries of K_a^+, K_a^-
J_a	2.1	$J_a^+ \cap J_a^-$
G_a^+, G_a^-	2.1	pluriharmonic functions that measure the rate of escape to infinity under forward (backward) iterates of f_a
α	2.4	parameter used in the definition of V_+, V_-

V_+, V_-	2.4	$\{\vert x\vert > \alpha, \vert x\vert > \vert y\vert\}$, $\{\vert y\vert > \alpha, \vert y\vert > \vert x\vert\}$ – regions which describe the large scale behavior of the Hénon map
W	2.4	$\{\vert x\vert \le \alpha, \vert y\vert \le \alpha\}$
D_R	2.4	the disk of radius R in the parameter space, used to define V_+ and V_-
$\varphi_{a,+}, \varphi_{a,-}$	2.4	holomorphic functions that semiconjugate dynamics in V_+, V_- to $z \to z^2$, $z \to z^2/a$
s_k^+, s_k^-	2.4	auxillary functions, used to study $\varphi_{a,+}$ and $\varphi_{a,-}$
$C(p)$	2.4	the curve $y = p(x)$, this is J_0^-
G_p	2.3	Green function for the map $x \to p(x)$
b_p	2.4	Böttcher coordinate for the map $x \to p(x)$
$\mathcal{F}_a^+, \mathcal{F}_a^-$	2.1	foliations of U_a^+, U_a^-
$\mathcal{C}_a$	2.1	the critical locus, the set of tangencies between foliations $\mathcal{F}_a^+$ and $\mathcal{F}_a^-$
(x_n, y_n)	2.4	$(x_n, y_n) = f_a(x, y)$
$\mathcal{D}_k^+$	2.4	the domain of definition of $\varphi_{a,+}^k$
$\mathcal{D}_k^-$	2.4	the domain of definition of $\varphi_{a,-}^k$
u		$u = p(y) - x$, measures the distance from a point (x, y) to $C(p)$
Ω	2.3	$\{G_a^+ \le r\} \cap \{\vert y\vert \le \alpha\} \cap \{\vert p(y) - x\vert < \alpha\vert a\vert\}$, the domain that does not intersect with its images under f_a and f_a^{-1}
$\hat{\Omega}_1$	2.13	$\{(x, y) \in \mathbb{CP}^2 \vert \quad \vert y\vert < \varepsilon, \vert x\vert \ge \alpha\}$

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