

# EXCITED RANDOM WALK AGAINST A WALL

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ABSTRACT. We analyze random walk in the upper half of a three dimensional lattice which goes down whenever it encounters a new vertex, a.k.a. excited random walk. We show that it is recurrent with an expected number of returns of  $\sqrt{\log t}$ .

## 1. INTRODUCTION

The model we will analyze in this paper (see section ?? for a precise definition) is a variation on excited random walk. Excited random walk is a walk on a  $d$ -dimensional lattice ( $d = 1$  seems to be the richest case) which has a drift in some fixed direction whenever it encounters a new vertex. See [?, ?, ?, ?, ?] for recent results, [?, ?] for a Brownian motion analog, and [?] for some simulation results. Excited random walk is proving to be far more tractable than other self interacting processes such as the reinforced random walk or the “true” self-avoiding walk.

In this paper we shall perform excited random walk on a half space. Thus the walk’s natural drift downward is counterbalanced by the stiff floor. In a sense, the walk exhibits a self critical behavior: if the walk “tries to escape” and visits a large number of new vertices, it is pushed down to the floor and becomes almost 2 dimensional and thus recurrent. If on the other hand the walk returns too many times to the same vertices, it will upon returning typically behave like simple random walk in 3 dimensions, which is transient.

Thus, if a two dimensional random walk has approximately  $\log t$  returns to the origin until time  $t$ , and a three dimensional random walk has approximately 1 such return, we should expect excited random walk to take some intermediate value. A somewhat less vague, but still heuristic argument, says that the value should be  $\sqrt{\log t}$ : the projection of the walk on the  $(x, y)$  plane is a two dimensional random walk so it returns to every column about  $\log t$  times. If it reaches  $x$  vertices in the column, it would accumulate a downward drift of  $x$ . Assuming homogeneous-ness, it would visit the floor about  $\log t/x$  times and accumulate this amount of upward drift. Since these should balance we get  $x = \log t/x$  or  $x = \sqrt{\log t}$ .

We shall prove that the  $\sqrt{\log t}$  heuristic is in fact accurate, and get in particular that the walk is recurrent, a fact which is not at all clear a-priori. The actual proof only follows the heuristic half way. The proof of the upper bound (see section ??) will use different methods. The proof of the lower bound will mimic the heuristic argument, but will use the already established upper bound. An important tool in proving the lower bound will be a coupling argument between two instances of excited random walk (section ??), which enables us to strengthen the upper bound and replace the role of homogeneous-ness in the heuristic. Unfortunately, we were forced to assume a deterministic downward drift to make ends meet.

As a side remark, the square root heuristic also works for the analogous model in two dimensions, and one gets that the average number of visits of excited random walk to the floor is of the order of  $\sqrt[4]{t}$ . The two dimensional case is less interesting because recurrence can be proved easily by coupling to simple random walk (so that the simple random walk is always higher than the excited random walk) and this argument does not require deterministic drift or specific floor behavior. We will not present any details of the two dimensional case.

**1.1. Open Problems.** As already remarked, we were not able to prove the case where the walk, upon hitting a new vertex, goes down with some probability  $p < 1$ . The upper bound (theorem ??) carries through unchanged, but the coupling argument (lemma ??) requires that the configuration be downward closed which is not true for a probabilistic drift meaning that our techniques only give a lower bound for the *average* number of visits to the floor, but one cannot deduce from that a lower bound for the number of visits to a specific vertex, or even recurrence.

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## 2. PRELIMINARIES

**Definition.** In this paper, excited random walk (ERW) is a process of points  $\{R(t) = (x(t), y(t), z(t))\}_{t=1}^{\infty}$  in  $\mathbb{H} := \{(x, y, z) \in \mathbb{Z}^3 : z \geq 0\}$  such that  $R(0) = \underline{0} = (0, 0, 0)$  and  $R(t+1)$  is created as follows:

- Floor — when the walk is currently on a floor vertex, i.e.  $z(t) = 0$  it moves with probability  $\frac{1}{5}$  up and with probability  $\frac{1}{5}$  to either of the 4 sides.
- Visited — when the walk is on a non-floor vertex it visited before, namely  $R(t) = R(u)$  for some  $u < t$ , then it moves like a simple random walk.
- New — when the walk is at a non-floor vertex it never visited, then it moves downwards deterministically, namely  $R(t+1) = (x(t), y(t), z(t)-1)$ .

We can also talk about an “ERW starting from  $v$ ” for some  $v \in \mathbb{H}$  and in this case we take  $R(0) = v$  instead.

A more-or-less equivalent process is the symmetric ERW, defined on  $\mathbb{Z}^3$  with the “excited steps”, i.e. the steps performed when reaching a new vertex  $(x, y, z)$  go down if  $z > 0$  and up if  $z < 0$ . Also, a vertex  $(x, y, z)$  is considered visited if  $R(u) = (x, y, z)$  or if  $R(u) = (x, y, -z)$  for some  $u < t$ . At the middle level the walk has probability  $\frac{1}{5}$  for the sides and  $\frac{1}{10}$  for the up or down. Thus if  $R(t)$  is a symmetric ERW then  $(x(t), y(t), |z(t)|)$  is an ERW, and vice versa, an ERW can be symmetrized by adding random coin flips that will decide, whenever the walk is at  $z = 0$  whether to go up or down.

Given an ERW  $R$  at some time  $t$ , we denote by  $\text{Vis}_R(t)$  the set of visited (non-floor) vertices i.e.  $\{R(u) : u < t\} \setminus \{(x, y, 0)\}$ . When  $t$  is clear from the context, we shall omit it, referring to the set of visited vertices as  $\text{Vis}_R$ .

It is important to notice that conditioning on the past (i.e. on  $R[0, t]$  for some  $t$ ) is identical to conditioning on  $\text{Vis}(t)$  and  $R(t)$ . This can be given a formal meaning using the notion of conditional probability (see e.g. [?, 4.1c]) but instead we shall use the following definition. For any set of vertices  $\mathcal{V} \subset \mathbb{H}$  we shall define an ERW “starting from  $(v, \mathcal{V})$ ” by defining  $R(0) = v$  and  $\text{Vis}_R(0) := \mathcal{V}$  and continuing in the natural way. Clearly,  $R[t, \infty)$  is the same as an ERW starting from  $(R(t), \text{Vis}_R(t))$ . Some of our results (mainly theorem ??) hold also for an ERW starting from  $\mathcal{V}$  for a general  $\mathcal{V}$ . In others (like theorem ??) this generalization requires assuming that  $\mathcal{V}$  is a legal configuration, i.e. a configuration which can be  $\text{Vis}_R(t)$  with positive probability. It is easy to see that this is equivalent to  $\mathcal{V}$  being finite, connected and downward-closed.

The *hitting time* of a random walk  $R$  (simple or excited) of a subset  $A \subset \mathbb{Z}^3$  is the first time  $t > 0$  such that  $R(t) \in A$ . Notice that due to the requirement  $t > 0$  the hitting time is non-trivial even if  $A$  contains the starting point  $R(0)$ .

For a subset  $A \subset \mathbb{Z}^3$  we denote by  $\partial A$  the internal boundary of  $A$  i.e. all vertices in  $A$  with a neighbor outside  $A$ .  $B(v, r)$  will denote a ball around  $v$  with radius  $r$ .

When we write  $\log n$  we always mean  $\max\{1, \log n\}$  and  $\log 0 := 1$ . We use  $C$  and  $c$  to denote various universal positive constants, which could take different values, even inside the same formula.  $C$  will be used for constants which are “large enough” and  $c$  for constants which are “small enough”.

### 3. UPPER BOUND ON THE NUMBER OF VISITS TO A POINT

In this section we shall prove the following:

**Theorem 1.** *Let  $R$  be an ERW starting from some point  $w \in \mathbb{H}$  and some configuration  $\mathcal{V}$ . Let  $v \in \mathbb{H}$ . Let  $V(t) = V(t; v)$  be the number of times  $R$  returns to  $v$  until time  $t$ . Then*

$$\mathbb{E}V(t) \leq C\sqrt{\log t}.$$

We shall use the theorem for  $\mathcal{V}$  empty or the visited set of some past ERW. It should be noted, however, that the theorem holds when starting from any configuration, even one which is impossible to achieve using ERW, for example an isolated visited vertex.

The proof requires dividing into subshells and doing careful analysis of the transitions of ERW from one shell to the next. This is somewhat technical, so we shall first (section ??) sketch a “single shell” (well, a ball) argument which gives a weaker result with some simplifying assumptions. We hope this makes the proof clearer. Next (section ??) we shall give some simple lemmas that allow to compare two one-dimensional processes. After that (section ??) we shall analyze ERW in a single shell, and finally (section ??) we shall wrap the whole thing up.

We shall prove the claim for a symmetric ERW, and lose only a factor of two in the constant  $C$ . Obviously, we may assume  $t$  is large enough. It will be convenient to assume  $V(t; w)$  also counts time 0.

**3.1. Simplified sketch.** The argument we will now sketch only gives that the number of returns to a vertex  $v$  is  $\leq C \log^{5/6} t$ . To see this, examine symmetric ERW and take a ball around  $v$  with radius  $\log^{1/3} t$  and examine “visits to the ball” by which we mean the time between one hitting of the ball and the next escaping from a ball of double radius. Inside the ball there are only  $\log t$  vertices therefore, if

we think about visits to the ball that pass through more than  $\log^{1/6} t$  new vertices as “bad”, then there can be no more than  $\log^{5/6} t$  bad visits. On the other hand, a “good” visit is limited by the amount it differs from a simple random walk: even if we allow an opponent with total view of past and future to distort a simple random walk at less than  $\log^{1/6} t$  times, she cannot force the walk to pass through  $v$  unless the original, simple random walk, passes through a ball of radius  $\log^{1/6} t$  around  $v$ , the probability for which (at every visit to the outer ball) is  $\leq C \log^{-1/6} t$ . Since, from two dimensional arguments, there are only  $\log t$  such visits, then the total number of good visits to  $v$  is also bounded by  $C \log^{5/6} t$ . Summing the good and the bad proves the claim.

### 3.2. One dimensional processes.

**Lemma 1.** *Let  $R$  be a nearest-neighbor stochastic process on  $\{1, \dots, n\}$  with a uniform bound*

$$\mathbb{P}(R(t) = R(t) + 1 \mid R[0, t]) \leq q_{R(t)} \quad \forall R(t) \in \{2, \dots, n-1\}$$

for some numbers  $\{q_2, \dots, q_{n-1}\}$ . Let  $S$  be a nearest-neighbor Markov chain on  $\{1, \dots, n\}$  starting from  $R(0)$  with the transition probabilities (again from  $i$  to  $i+1$ ) equal to  $q_i$ . Let  $s$  and  $r$  be the probabilities that  $S$  and  $R$  respectively hit  $n$  before hitting 1. Then

$$r \leq s.$$

*Proof.* This follows directly from the fact that we can couple the two processes in a way such that  $S$  is always to the right of  $R$  and their difference is always even. See e.g. [?] for some background on the coupling method.  $\square$

**Lemma 2.** *Assume  $q_i \leq q < \frac{1}{2}$ ,  $i = 2, \dots, n-1$  are the transition probabilities of a nearest-neighbor Markov chain  $R$  on  $\{1, \dots, n\}$  and let  $r_j$  be the probability that  $R$  hits  $n$  before hitting 1 if  $R$  starts at  $j$ . Then*

$$r_{j+1} \geq r_j(1 + c)$$

where the constant  $c > 0$  may depend on  $q$  but not on  $n$ .

*Proof.* Let  $S$  be a nearest-neighbor Markov chain on  $\{1, \dots, j+1\}$  starting from  $j$  with transition probabilities  $q$ . Then the probability that  $S$  reaches 1 before  $j+1$  can be calculated explicitly [?, example 5.3.5] and is  $> (1 - 2q)/(1 - q)$  which we may denote by  $c$ . Hence (using lemma ??) we get that the probability  $p$  of  $R$  starting from  $j$  to reach 1 before  $j+1$  satisfies  $p > c$ . However,

$$r_j = (1 - p)r_{j+1}$$

and we are done.  $\square$

**Lemma 3.** *Assume  $0 < p \leq q_i \leq q < \frac{1}{2}$  are the transition probabilities of a nearest-neighbor Markov chain  $R$  on  $\{1, \dots, n\}$  starting from  $j$  and let  $r_j$  be the probability that  $R$  hits  $n$  before hitting 1. Then*

$$r_j \leq C \left( \frac{p}{1-p} \right)^{n-j} \prod_{i=1}^n (1 + C(q_i - p))$$

where  $C$  may depend on  $p$  and on  $q$  but not on  $n$  or on the  $q_i$ -s.

*Proof.* In the case  $q_i = p$  for all  $i$ , the numbers  $r_j$  satisfy a simple quadratic recursion, namely  $r_{j+2} - \frac{1}{p}r_{j+1} + \frac{1-p}{p}r_j = 0$ , which can be solved explicitly to show that  $r_j \leq C(p/(1-p))^{n-j}$ . Hence it is enough to measure the effect of a change in one  $q_i$  namely, to show that if  $q'_i = q_i$  except for one  $I$ , and  $q'_I > q_I$  then

$$r'_j \leq r_j(1 + C(q'_I - q_I)). \quad (1)$$

Let  $A$  and  $B$  be some parameters. The numbers

$$s_j := \begin{cases} r_j & j < I \\ Ar_j + B & j \geq I \end{cases}$$

are ‘‘harmonic’’ except possibly at  $I - 1$  and  $I$  (meaning that  $s_j = q'_j s_{j+1} + (1 - q'_j) s_{j-1}$ ) hence for the values of  $A$  and  $B$  satisfying that  $s_I = r_I$  and  $s_I = q'_I s_{I+1} + (1 - q'_I) s_{I-1}$  we would get that  $s_i$  is harmonic and as a consequence,  $r'_j \equiv s_j / s_n$ . These conditions give that for the required  $s_{I+1}$ ,

$$|s_{I+1} - r_{I+1}| \leq r_I \left| \frac{1}{q'_I} - \frac{1}{q_I} \right| + r_{I-1} \left| \frac{1}{1 - q'_I} - \frac{1}{1 - q_I} \right|$$

and by lemma ?? this gives  $|s_{I+1} - r_{I+1}| \leq Cr_I(q'_I - q_I)$ . Hence ( $A$  and  $B$  are linear in  $s_I$  and  $s_{I+1}$ ) we get that

$$A = 1 + O(q'_I - q_I) \frac{r_I}{r_{I+1} - r_I} \quad B = O(q'_I - q_I) \frac{r_I^2}{r_{I+1} - r_I}$$

(the constant implicit in the notation  $O$  here may also depend on  $p$  and  $q$ ). Another appeal to lemma ?? allows to replace the  $r_I/(r_{I+1} - r_I)$  factors with a constant and we get that  $s_n = 1 + O(q'_I - q_I)$ . Hence we get (??) and the lemma.  $\square$

**3.3. Behavior in shells.** The proof of theorem ?? in the next section uses lemma ?? for one-dimensional processes created by examining the hitting times of shells of radius  $4^n$ . A three dimensional Brownian motion (which is our model, in some vague sense) starting from a point on a shell of radius  $r$  has probability  $\frac{1}{5}$  to reach  $\frac{1}{4}r$  before  $4r$ , independently of the starting point. For an ERW this probability depends on the starting point and on  $\text{Vis}$ , but we will compare the probability that a *good visit*, in the same sense as in the proof sketch above, reaches  $\frac{1}{4}r$  before  $4r$  to  $\frac{1}{5}$ . This will be done in lemma ?. The other lemmas handle boundary cases: lemmas ?? and ?? are for the outermost shell and lemma ?? is for the innermost shell.

**Lemma 4.** *Let  $r \geq 1$  and let  $K_r := B((x, y), r) \times \mathbb{Z}$  namely an infinite vertical cylinder around  $v = (x, y, z)$ . Define stopping times  $t_i^{\text{in/out}}$  as follows:  $t_0^{\text{out}} := 1$ , and for  $i \geq 1$ ,*

$$\begin{aligned} t_i^{\text{in}} &:= \min\{t \geq t_{i-1}^{\text{out}} : R(t) \in K_r\} \\ t_i^{\text{out}} &:= \min\{t > t_i^{\text{in}} : R(t) \notin K_{2r}\}. \end{aligned} \quad (2)$$

Define  $l := \max\{i : t_i^{\text{in}} < t\}$ , ‘‘the number of visits to  $K$ ’’. Then

$$\mathbb{P}(l > \lambda \log t) \leq Ce^{-c\lambda} \quad \forall \lambda > 0.$$

*Proof.* For every  $i > 1$  we know that  $t_{i-1}^{\text{out}}$  is in the exterior boundary of  $K_{2r}$  which we denote by  $S$ . Hence for  $i > 1$

$$\mathbb{P}(t_i^{\text{in}} > t_{i-1}^{\text{out}} + t) \geq \min_{w \in S} \mathbb{P}^w(R \text{ hits } \partial K_{t+2r} \text{ before } K_r)$$

which is a purely two-dimensional question. Denoting by  $a$  the two-dimensional discrete harmonic potential we can continue the inequality (see [?] for a nice exposition of the connection between harmonic functions and the harmonic potential in particular and hitting probabilities)

$$\geq \frac{\min\{a(w) : w \in S\} - \max\{a(w) : w \in \partial K_r\}}{\max\{a(w) : w \in \partial K_{t+2r}\} - \min\{a(w) : w \in \partial K_r\}}$$

and since  $a(w) = A \log |w| + B + o(1)$  [?, P12.3]

$$= \frac{\log 2 + o(1)}{\log((t+2r)/r) + o(1)} \geq \frac{c}{\log t}$$

which gives the lemma immediately.  $\square$

**Lemma 5.** *Let  $r > 4$ , let  $v \in \mathbb{Z}^3$  and let  $w \in \partial B(v, r)$ . Let  $R$  be a (symmetric) ERW starting from  $w$  and some configuration  $\mathcal{V} \subset \mathbb{Z}^3$ , and let  $T$  be its hitting time of  $\partial B(v, 4r) \cup \partial B(v, \frac{1}{4}r)$ . Let  $\epsilon > r^{-1/2}$  be some parameter. Let  $G$  be the event that  $R$  encounters less than  $\epsilon r$  new sites until  $T$ . Then*

$$\mathbb{P}(\{R(T) \in \partial B(v, \frac{1}{4}r)\} \cap G) \leq \frac{1}{5} + C\epsilon^{1/4}. \quad (3)$$

*Proof.* Couple  $R$  to a simple random walk  $W$  in the following manner: if  $R$  is in a visited (non-floor) vertex let  $R$  and  $W$  perform the same step. Otherwise they walk independently according to their respective rules. The lemma will be mostly proved once we estimate  $|R(t) - W(t)|$ . There are two sources for the discrepancy: new vertices and floor vertices. Therefore let us write  $|R(t) - W(t)| = |N(t) + F(t)|$  where

$$N(t) = \sum_{\substack{u < t \\ R(u) \text{ is a new vertex}}} R(u+1) - R(u) - W(u+1) + W(u)$$

and  $F$  is the same for floor vertices. Now,  $G$  obviously implies  $|N(t)| \leq 2\epsilon r$  for all  $t \leq T$  so we need only estimate  $F$ . Now, for every time  $t$  when  $R$  is in a floor vertex, the expected motion of  $R$  is zero (remember that we are talking about the symmetric ERW). In other words, if we denote by  $t_i$  the  $i$ 'th hitting of the floor then  $F(t_i)$  is a symmetric random walk on  $\mathbb{Z}^3$  with bounded steps. By the reflection principle (see e.g. [?, chapter 2, lemma 1])  $\max_{i \leq n} |F(t_i)|$  has the same tail behavior as  $|F(t_n)|$  i.e. a square-exponential one. Denoting by  $f$  the number of times  $R$  hits the floor by time  $T$  we get

$$\mathbb{P}(\{\max_{t \leq T} |F(t)| > \lambda \sqrt{N}\} \cap \{f \leq N\}) \leq Ce^{-c\lambda^2} \quad \forall \lambda > 0, \forall N \in \mathbb{N}. \quad (4)$$

Examine one time  $t$  when  $R(t)$  is at the floor. It is easy to see that a simple random walk starting from a floor point has probability  $\geq c/r$  to exit the ball  $B(v, 4r)$  before returning to the floor. Therefore we have

$$\mathbb{P}(R \text{ hits } \partial B(v, 4r) \text{ or some new vertex before returning to the floor}) \geq \frac{c_1}{r}.$$

This implies that if  $f > (2/c_1)\epsilon r^2$  then with probability  $> 1 - Ce^{-c\epsilon r}$  there are at least  $\epsilon r + 1$  times  $t_i < T$  when the event above happened. This, however, contradicts the event  $G$  so we get

$$\mathbb{P}(\{f > (2/c_1)\epsilon r^2\} \cap G) \leq Ce^{-c\epsilon r}. \quad (5)$$

Combining (??) and (??) we get

$$\mathbb{P}(\{\max_{t \leq T} |F(t)| > C\lambda r\sqrt{\epsilon}\} \cap G) \leq Ce^{-c\lambda^2} + Ce^{-c\epsilon r} \quad \forall \lambda > 0.$$

We pick  $\lambda := \frac{1}{C}\epsilon^{-1/4}$  and get (using also the requirement  $\epsilon > r^{-1/2}$ )

$$\mathbb{P}(\{\max_{t \leq T} |F(t)| > r\epsilon^{1/4}\} \cap G) \leq C\epsilon.$$

This is the estimate of  $F$  that we need.

Now, in general we have for any  $s \geq 1$  that  $R(T) \in \partial B(v, \frac{1}{4}r)$  implies that either for some  $t \leq T$  we have  $|N(t) + F(t)| > s$  or that  $W$  hits  $\partial B(v, \frac{1}{4}r + s)$  before hitting  $\partial B(v, 4r + s)$ . The probability for that to happen (denote it by  $q$ ) we calculate using the discrete Green function of  $\mathbb{Z}^3$  (denote it by  $a$ ) the same way we used the harmonic potential in the previous lemma. Since  $a(z) = \frac{c}{|z|} + O\left(\frac{1}{|z|^3}\right)$  [?, Theorem 4.3.1] we get  $q = \frac{1}{5} + O(s/r)$ . Applying this with  $s = r(2\epsilon + \epsilon^{1/4})$  we get

$$\begin{aligned} \mathbb{P}(\{R(T) \in \partial B(v, \frac{1}{4}r)\} \cap G) &\leq \\ &\leq \mathbb{P}(W \text{ hits } \partial B(v, \frac{1}{4}r + s)) + \mathbb{P}(\{\max_{t \leq T} |N(t) + F(t)| > s\} \cap G) \leq \\ &\leq \frac{1}{5} + O(s/r) + C\epsilon \leq \frac{1}{5} + C\epsilon^{1/4}. \quad \square \end{aligned}$$

**Lemma 6.** *Let  $r > 4$ , let  $v = (x, y, z)$  and let  $w \in \partial B(v, r)$ . Let  $R$  be a (symmetric) ERW starting from  $w$  and some configuration  $\mathcal{V} \subset \mathbb{Z}^3$ , and let  $T$  be its hitting time on  $\partial B(v, 4r) \cup \partial B(v, \frac{1}{4}r)$ . Denote*

$$Q := (\partial B(v, 4r) \cap B((x, y), 2r) \times \mathbb{Z}) \cup \partial B(v, \frac{1}{4}r)$$

*i.e.  $Q$  is the union of a) the outer sphere intersected with a concentric vertical cylinder of half its radius and b) the inner sphere. Let  $G$  be the event that  $R$  encounters less than  $\epsilon r$  new sites until  $T$  for some  $\epsilon$  sufficiently small. Then*

$$\mathbb{P}(\{R(T) \in Q\} \cap G) \leq 1 - c$$

The proof is very similar to the proof of the previous lemma — in fact, simpler — the only additional fact needed is that a *simple* random walk has probability  $< 1 - c$  to hit  $Q$ . This is quite easy to see and we omit any further details about the proof of lemma ??.

**Lemma 7.** *Let  $r > 1$ , let  $v \in \mathbb{Z}^3$  and let  $w \in \partial B(v, r)$ . Let  $R$  be a (symmetric) ERW starting from  $w$  and some configuration  $\mathcal{V} \subset \mathbb{Z}^3$ , and let  $T$  be its hitting time of  $\partial B(v, 4r) \cup \{v\}$ . Let  $G$  be the event that  $R$  encounters no new sites until  $T$ . Then*

$$\mathbb{P}(\{R(T) = v\} \cap G) \leq C/r. \quad (6)$$

*Similarly if  $R$  starts from  $v$  then this probability is  $\leq 1 - c$ .*

*Proof.* This time we couple  $R$  to a random walk  $W$  which has the same behavior as  $R$  at the floor, i.e. when  $W$  hits the floor it has probability  $\frac{1}{5}$  to go to each of its floor neighbors, and probability  $\frac{1}{10}$  for each of its vertical neighbors, but other than that is simple. Clearly if  $G$  happened then  $R(t) = W(t)$  for all  $t$  so it is enough to estimate the corresponding probabilities for  $W$ .

However  $W$  is a reversible random walk (meaning that it can be realized as a walk on a weighted graph) so Varopoulos [?] and Hebisch and Saloff-Coste [?,

theorem 2.1] apply. Together they give that the probability  $p_t(x, y)$  that  $W$  starting from  $x$  and going  $t$  steps will be at  $y$  satisfies

$$p_t(x, y) \leq \frac{C}{t^{3/2}} e^{-c|x-y|^2/t} \quad \forall x, y, t.$$

Summing over  $t$  gives that the discrete Green function  $a(x, y) = \sum_t p_t(x, y)$  satisfies  $a(x, y) \leq C/|x - y|$  and  $a(x, x) \leq C$ .  $a$  is harmonic so the same calculations as in lemma ?? give the estimates for the probabilities.  $\square$

**3.4. Proof of theorem ??.** Let  $\beta_j = 4^{-j} \sqrt{\log t}$  defined for  $j = 1, \dots, J$  until the first  $J$  such that  $\beta_J < \sqrt[9]{\log t}$ , and let  $\beta_{J+1} = \beta_{J+2} = 1$ . Let  $S_i := \partial B(v, \beta_i)$ , and in particular  $S_{J+1} = S_{J+2} = \{v\}$ . The spheres  $S_i$  are the analogue of the sphere at  $\log^{1/3} t$  discussed in the “simplified sketch” section. Let  $t_i$  denote stopping times at these spheres defined, somewhat similarly to (??), by

$$t_1 = \min u : R(u) \in \bigcup_{j=1}^{J+1} S_j$$

$$t_{i+1} = \min u > t_i : \begin{cases} u \in S_{j-1} \cup S_{j+1} & \text{when } R(t_i) \in S_j, 2 \leq j \leq J+1 \\ u \in S_2 & \text{when } R(t_i) \in S_1 \end{cases}$$

(notice the asymmetry at  $v$  — the only case where  $R(t_i)$  and  $R(t_{i+1})$  may belong to the same  $S_j$ ). Let  $\delta \in (0, 1)$  be some parameter sufficiently small to be fixed later. In fact, it is enough to take  $\delta := \min\{(2/15C_{\text{lemma ??}})^4, \epsilon_{\text{lemma ??}}, \frac{1}{2}\}$ , but the only meaning of this expression is in the various conditions that will appear below. Let  $G_i$  ( $G$  standing for “good”) be the event that  $R$  hits less than  $\delta j^{-8} \beta_{j+1}$  new sites between time  $t_i$  and time  $t_{i+1}$  where  $j \leq J+1$  is given by  $R(t_i) \in S_j$ . Obviously, there is nothing stopping  $\delta j^{-8} \beta_{j+1}$  to be smaller than 1 (indeed it must be if  $j \geq J$ ). Remembering the “simplified sketch” section, the event  $G_i$  is the analogue of the event “good visit to  $v$ ” with respect to the relevant sphere  $S_i$ .

To estimate  $V(t)$  we examine the walk performed before the time when  $v$  was hit, and ask: when has  $G_i^c$  (the complement of  $G_i$ ) occurred last? More precisely, define the event  $H_i^0$  to be  $G_i \cap G_{i+1} \cap \dots \cap G_{k-1}$  where  $k \geq i$  is the first such that  $R(t_k) \in S_1 \cup \{v\}$  and let  $H_i$  be the event that  $H_i^0$  happened and  $R(t_k) = v$ . In particular, if  $R(t_i) = v$  then  $H_i$  happens while if  $R(t_i) \in S_1$  it does not. Define

- (1)  $E_j$  ( $2 \leq j \leq J+1$ ) to be the number of  $i$ -s satisfying that  $R(t_i) \in S_j$ , that  $G_i$  did *not* happen, and that  $H_{i+1}$  did happen.
- (2)  $E_{J+2}$  to be the number of  $i$ -s such that  $R(t_i) = v$  and  $G_i \cap H_{i+1}$  happened.
- (3)  $E_1$  to be the number of  $i$ -s such that  $R(t_i) \in S_1$  and  $H_{i+1}$  happened.

For all these we count only  $i$ -s that the relevant  $t_k$ , i.e. the time where  $R(t_k) = v$ , happened before time  $t$ . Clearly,  $V(t) = \sum_{j=1}^{J+2} E_j$  therefore it is enough to estimate these  $E_j$ -s. Now if  $t$  is sufficiently large then we can apply lemma ?? for all  $j < J$  (the problem is only in the condition “ $\epsilon > r^{-1/2}$ ” of lemma ?? where here  $\epsilon = \frac{1}{4} \delta j^{-8}$  and  $r = \beta_j$ ). For  $j = J, J+1$  we use lemma ?? and in total we get

$$\mathbb{P}(G_i \cap \{R(t_{i+1}) \in S_{j+1}\} | R[0, t_i], R(t_i) \in S_j) \leq \begin{cases} \frac{1}{5} + C\delta^{1/4} j^{-2} & 1 \leq j < J \\ C \log^{-1/6} t & j = J \\ 1 - c & j = J+1. \end{cases} \quad (7)$$

Here and below we use the notation  $\mathbb{P}(X|Y, E)$  for a variable  $Y$  and an event  $E$  to mean the function  $\mathbb{P}(X|Y)$  restricted to  $E$  — here everything is discrete so this simply means that the inequality holds for any value of  $R[0, t_i]$  for which  $R(t_i) \in S_j$ . Denote the values on the right hand side by  $q_j$ . This allows us to estimate  $\mathbb{P}(H_i | R[0, t_i])$  by comparing the process  $R(t_i)$  to a Markov chain  $Q_j$  on  $\{1, \dots, J+1\}$  starting from  $j$  with the transition probabilities  $q_j$  (we use here lemma ??). If  $\delta$  is sufficiently small (explicitly if  $\delta \leq (2/15C_{\text{lemma}})^4$ ) we would have  $q_j \leq \frac{1}{3}$  for all  $j < J$ . Hence we can use lemma ?? on the interval  $[1, J]$  and we get

$$\mathbb{P}(Q_j \text{ hits } J \text{ before } 1) \leq C4^{j-J} \prod_{j=1}^J (1 + Cj^{-2}) \leq C4^{j-J} = C \frac{\beta_J}{\beta_j}.$$

The step from  $J$  to  $J+1$  contributes another  $C\beta_J^{-1}$  factor so we end up with

$$\mathbb{P}(H_i | R[0, t_i]) \leq \frac{C}{\beta_j} \quad \text{when } R(t_i) \in S_j, j \leq J \quad (8)$$

$$\mathbb{P}(G_i \cap H_{i+1} | R[0, t_i]) \leq 1 - c \quad \text{when } R(t_i) = v. \quad (9)$$

We note that  $B(v, \beta_{j-1}) \setminus B(v, \beta_{j+1})$  contains  $< C\beta_{j-1}^3$  points. Therefore the number of  $i$ -s such that  $G_i^c$  can occur together with  $R(t_i) \in S_j$  is no more than

$$\begin{aligned} \frac{C\beta_{j-1}^3}{\delta j^{-8}\beta_{j+1}} &\leq Cj^8 4^{-2j} \log t \quad \text{when } j < J \\ \beta_{j-1}^3 &\leq C\sqrt{\log t} \quad \text{when } j = J, J+1. \end{aligned} \quad (10)$$

(here and below we will be “folding” the  $\delta$  into the constants  $c, C$ ). Using (??) for  $i+1$  (and  $j+1$ , which would also estimate the case that  $R(t_{i+1}) \in S_{j-1}$ ) shows that  $E_j$  is dominated by a sum of independent Bernoulli trials with probability  $C\beta_{j+1}^{-1}$ , so

$$\mathbb{P}(E_j > Cj^8 4^{-j} \sqrt{\log t} + \lambda j^4 2^{-j} \log^{1/4} t) \leq Ce^{-c\lambda} \quad 2 \leq j < J \quad (11)$$

while for  $j = J, J+1$  we have deterministically

$$E_j \leq C\sqrt{\log t}. \quad (12)$$

In particular,  $\mathbb{E} \sum_{j=2}^{J+1} E_j \leq C\sqrt{\log t}$ .

Next we estimate  $E_1$ . We start with an estimate of the number of  $i$ -s such that  $R(t_i) \in S_2$ . Denote it by  $F$ . Define  $Q = (S_1 \cap K_{2\beta_2}) \cup S_3$  where  $K$  is an infinite cylinder as in lemma ?. By lemma ??, if only  $\delta$  is sufficiently small ( $\delta \leq \epsilon_{\text{lemma}}$ ),

$$\mathbb{P}(G_i \cap \{R(t_{i+1}) \in Q\} | R[0, t_i], R(t_i) \in S_2) \leq 1 - c$$

Hence the number  $X_1$  of times this event happened satisfies

$$\mathbb{P}(X_1 > (1 - c)F + \lambda\sqrt{F \log F}) \leq Ce^{-c\lambda^2}. \quad (13)$$

To prove (??) compare to an infinite sequence of Bernoulli trials  $\epsilon_i$  with probability  $1 - c$  for which a rough estimate (by summing over  $s$ ) shows that  $\mathbb{P}(\exists s \leq t : \sum_{i=1}^s \epsilon_i > (1 - c)s + \lambda\sqrt{s \log s}) \leq Ce^{-c\lambda^2}$ .

Next, the number of times  $G_i^c$  happened is bounded using (??) by  $C \log t$  so we get that the number  $X_2$  of  $i$ -s for which  $R(t_i) \in S_2$  and  $R(t_{i+1}) \notin Q$  satisfies

$$\mathbb{P}(X_2 \leq cF - C \log t - \lambda\sqrt{F \log F}) \leq Ce^{-c\lambda^2}. \quad (14)$$

However, every such event is an “entry into  $K_{\beta_2}$ ” in the sense of lemma ?? so

$$\mathbb{P}(X_2 > \lambda \log t) \leq C e^{-c\lambda}. \quad (15)$$

We get

$$\begin{aligned} \mathbb{P}(F > \lambda \log t) &\leq \mathbb{P}(X_2 > \sqrt{\lambda} \log t) + \mathbb{P}(\{F > \lambda \log t\} \cap \{F > \sqrt{\lambda} X_2\}) \leq \\ &\stackrel{(?)}{\leq} C e^{-c\sqrt{\lambda}} + \mathbb{P}(\{F > \lambda \log t\} \cap \{F > \sqrt{\lambda} X_2\}). \end{aligned} \quad (16)$$

For  $\lambda$  sufficiently large we have that  $F > \lambda \log t$  and  $F > \sqrt{\lambda} X_2$  imply that in fact  $F > C X_2 + C \log t + C \sqrt{\lambda F \log F}$  and so by (??)

$$\mathbb{P}(\{F > \lambda \log t\} \cap \{F > \sqrt{\lambda} X_2\}) \leq C e^{-c\sqrt{\lambda}}. \quad (17)$$

But (??) can be made to hold not just for  $\lambda$  sufficiently large by increasing the  $C$  on the right hand side and with (??) we get

$$\mathbb{P}(F > \lambda \log t) \leq C e^{-c\sqrt{\lambda}} \quad \forall \lambda > 0. \quad (18)$$

This is the estimate of  $F$  that we need.

On the other hand, let  $i$  satisfy the requirements for  $E_1$ , namely  $R(t_i) \in S_1$  and  $H_{i+1}$  has occurred. Using (??) we get

$$\begin{aligned} \mathbb{P}(H_{i+1} | R[0, t_i], R(t_i) \in S_1) &= \mathbb{E}\mathbb{P}(H_{i+1} | R[0, t_{i+1}], R(t_i) \in S_1) \leq \\ &\stackrel{(?)}{\leq} \mathbb{E}(C/\beta_2) = C/\beta_2 \end{aligned}$$

where  $\mathbb{E}$  here is the conditional expectation over the variable  $R[0, t_i]$ . As in (??) above, we get

$$\mathbb{P}(E_1 > C F/\beta_2 + \lambda \sqrt{(F/\beta_2) \log F}) \leq C e^{-c\lambda}. \quad (19)$$

Combining (??) and (??) gives

$$\begin{aligned} \mathbb{P}(E_1 > \lambda \sqrt{\log t}) &\leq \mathbb{P}(\{E_1 > \sqrt{\lambda}(F/\beta_2)\} \cap \{E_1 > \lambda \sqrt{\log t}\}) + \\ &\quad + \mathbb{P}(F/\beta_2 > \sqrt{\lambda \log t}) \leq C e^{-c\lambda^{1/4}}. \end{aligned} \quad (20)$$

Hence  $\mathbb{E}E_1 \leq C \sqrt{\log t}$  and this part is estimated as well.

The estimate of  $E_{J+2}$  comes from (??): again by comparing to a sum of independent Bernoulli trials we get

$$\mathbb{P}(E_{J+2} \geq (1-c)V(t) + \lambda \sqrt{V(t) \log V(t)}) \leq C e^{-c\lambda^2}$$

or, equivalently,

$$\mathbb{P}(E_{J+2} \geq C(V(t) - E_{J+2}) + \lambda \sqrt{V(t) \log V(t)}) \leq C e^{-c\lambda^2}.$$

Adding (??), (??) and (??) gives

$$\mathbb{P}(V(t) - E_{J+2} \geq \lambda \sqrt{\log t}) = \mathbb{P}\left(\sum_{j=1}^{J+1} E_j \geq \lambda \sqrt{\log t}\right) \leq C e^{-c\lambda^{1/4}}$$

so

$$\begin{aligned} \mathbb{P}(E_{J+2} \geq \lambda \sqrt{\log t}) &\leq \mathbb{P}(\{E_{J+2} \geq \sqrt{\lambda}(V(t) - E_{J+2})\} \cap \{E_{J+2} \geq \lambda \sqrt{\log t}\}) + \\ &\quad + \mathbb{P}(V(t) - E_{J+2} \geq \sqrt{\lambda \log t}) \leq C e^{-c\lambda^{1/8}} \end{aligned} \quad (21)$$

which shows that  $\mathbb{E}E_{J+2} \leq C\sqrt{\log t}$  and since this is the last term in  $V(t)$ , the theorem is proved.  $\square$

**Corollary 1.** For every  $v \in \mathbb{H}$  we have

$$\mathbb{E}(V(t; v) | V(t; v) \neq 0) \leq C\sqrt{\log t} \quad (22)$$

where  $V(t; v)$  is the number of visits to  $v$  after  $t$  steps.

*Proof.* This is because conditioning on  $V(t; v) \neq 0$  is identical to considering an unconditioned ERW starting from  $(v, \mathcal{V})$  where  $\mathcal{V} := \text{Vis}(T)$ ,  $T$  being the hitting time of  $v$  and walking for a distance of  $t - T$ . Applying theorem ?? shows that  $\mathbb{E}V(t - T; v) \leq C\sqrt{\log t}$  for any  $\mathcal{V}$  and integrating over  $T$  and  $\mathcal{V}$  shows (??).  $\square$

**Corollary 2 (exponential decay of  $V$ ).** There exist constants  $c, C$  s.t. for any point  $v$  and any  $\lambda > 0$ ,  $\mathbb{P}(V(t; v) > \lambda\sqrt{\log t}) < Ce^{-c\lambda}$ .

*Proof.* Using theorem ?? and Markov's inequality we get some constant  $K$  such that for every configuration  $\mathcal{V}$  one has that an ERW starting from  $(v, \mathcal{V})$  has probability  $< \frac{1}{2}$  to visit  $v$  more than  $K\sqrt{\log t}$  visits in the next  $t$  steps. Define  $L := \lfloor K\sqrt{\log t} \rfloor + 1$  and let  $\tau_k$  be the  $kL$ 'th return to 0 (here  $\lfloor \cdot \rfloor$  stands for the integer value). As in the previous corollary, the ERW after  $\tau_k$  is the same as an ERW starting from  $(v, \text{Vis}(\tau_k))$  so we get

$$\mathbb{P}(\tau_{k+1} > \tau_k + t | R[0, \tau_k]) > \frac{1}{2}.$$

Hence we get that  $\mathbb{P}(\tau_k < t) \leq 2^{-k}$ , which was to be proved.  $\square$

**3.5. Postfix remarks.** The values chosen for the  $\beta_j$  are in some sense “non-optimal”. A more natural choice would be  $\beta_j = e^{-2^j} \sqrt{\log t}$ , i.e. a doubly exponential decreasing sequence. For example, if one decides to use only a finite number of  $\beta$ -s (finite in the sense that the length  $J$  is independent of  $t$ ) and looks for the optimal  $\beta$ -s, the optimality requirement gives a set of equations which, when solved, give a doubly exponential decreasing sequence with  $\beta_1 = \log^{1/2-\epsilon} t$  and  $\beta_J = \log^{1/6} t$ . Actually, the fact that we stopped our series  $\beta_j$  when reaching  $\sqrt[6]{\log t}$  is an atavism from this optimization. Either choice for the  $\beta_j$  would give the same conclusion in the theorem.

Lemmas ??, ?? and ?? could have been simplified significantly if the behavior of the ERW at the floor would have been  $\frac{1}{6}$  for its floor neighbors and  $\frac{1}{3}$  for its upper neighbor. Unfortunately, the coupling argument used in the next section requires the probability of the upper neighbor to be  $\leq \frac{1}{5}$ .

Since corollary ?? gives a very simple argument for the exponential decay of  $V$ , one might wonder why did we bother with all the intermediate estimates of the form  $\mathbb{P}(\text{something} > \lambda\sqrt{\log t}) \leq C \exp(-c\lambda^{\text{some fraction}})$ , namely (??), (??) or (??)? They seem to be necessary for the calculation of  $E_{J+2}$ , (??). We would like to see a proof that can estimate  $\mathbb{E}E_{J+2}$  using only  $\sum_{j=1}^{J+1} \mathbb{E}E_j$ , but we were not able to overcome some dependency issues.

**Conjecture.** The correct tail decay is square-exponential, namely  $\mathbb{P}(V(t) > \lambda\sqrt{\log t}) \leq Ce^{-c\lambda^2}$ .

One possible interpretation of the word “correct” above is: for every  $\lambda$  and every  $t > t_0(\lambda)$ ,  $\mathbb{P}(V(t) > \lambda\sqrt{\log t}) > ce^{-C\lambda^2}$ .

## 4. THE COUPLING ARGUMENT

As we will show below, when the starting configurations are downward closed, it is possible to couple two instances of ERW such that one is always above the other. Here it is more convenient to think about them as walks in a half space rather than as the symmetrized version we used in the previous chapter, so from now on we will use the half space version of ERW. The following lemma uses this argument to show a certain monotonicity in the hitting probabilities. It will be crucial towards the end.

**Lemma 8.** *Let  $R, S$  be two ERWs, starting from a  $w \in \mathbb{H}$ , and from visited configurations satisfying  $\text{Vis}_R(0) \subset \text{Vis}_S(0)$  which are both downward-closed. Let  $v \in \mathbb{H}$  be a floor vertex and let  $V_R(t)$  and  $V_S(t)$  be the number of visits of  $R$  and respectively  $S$  to  $v$  in the first  $t$  steps. Then for any  $t \in \mathbb{N}$  and  $k \in \mathbb{N}$  we have  $\mathbb{P}(V_R(t) \geq k) \geq \mathbb{P}(V_S(t) \geq k)$ , and in particular  $\mathbb{E}(V_R(t)) \geq \mathbb{E}(V_S(t))$ .*

*Proof.* We define a coupling between  $R$  and  $S$  so that for any instance of the coupling the number of times  $R$  hits  $v$  before time  $t$  is greater or equal to the number of times  $S$  hits  $v$  before time  $t$ . The coupling requires a time change so, if we denote by  $\tau$  the number of coupling steps, we need two time change functions  $t_R(\tau)$  and  $t_S(\tau)$  to get back the time for each process. For brevity, we will replace  $R(t_R(\tau))$  with just  $R(\tau)$  or just with  $R$  (ditto for  $S$ ).

To define the coupling recall the three types of vertices an ERW can be at: floor, visited and new. We define the coupling according to the types of the vertices both walks are at, generally trying to make them walk “together”:

- If both  $R$  and  $S$  are at the same type of vertex — they move together (i.e. make the same step).
- If one of them is at a new vertex, and the other is not — the one at the new vertex makes a move downwards, while the other one waits.
- If one of them is at a visited vertex, and the other at a floor vertex, we let the first one move. If the move it made was downwards — the second walk waits. Otherwise, the second walk moves in the same direction.

We denote by  $\text{Wait}_R(\tau)$  ( $\text{Wait}_S(\tau)$ ) the number of times  $R$  ( $S$ ) waited until time  $\tau$  of the coupling. Thus  $t_R(\tau) = \tau - \text{Wait}_R(\tau)$  that is until step  $\tau$  of the coupling the walk  $R$  makes  $\tau - \text{Wait}_R(\tau)$  real steps.

As above, when looking at a specific step  $\tau$  of the coupled walk, we omit the step index from the various values thus writing  $x_R$  and  $\text{Wait}_R$  instead of  $x_R(\tau)$  and  $\text{Wait}_R(\tau)$ .

The lemma will now follow from the following claim:

**Claim.** *At each step of the coupling we have:*

- (1)  $x_R = x_S$  and  $y_R = y_S$ .
- (2)  $z_R - \text{Wait}_R = z_S - \text{Wait}_S$ .
- (3)  $z_R \leq z_S$ .
- (4) *If  $S$  is not at a new vertex, then  $\text{Vis}_S$  is downward closed and  $\text{Vis}_R \subset \text{Vis}_S$ .*

*Proof.* First notice that both walks make the same moves on the  $(x, y)$ -plane, regardless of the vertex type they are at, so at each step  $x_R = x_S$  and  $y_R = y_S$ , giving (1).

(2) follows from the fact that at each step of the coupling the walks either move together, or one of them waits while the other moves down.

To prove items (3) and (4) we use induction on the step of the coupling. Assume that the claim holds up some step, and look at the next step of the coupling.

We will first prove item (3) continues to hold. If both walks make the same move, (3) continues to hold. Otherwise, we are in one of the following two situations:

- (1) One of the walks is on a floor point, and the other one is above it.
- (2) One of the walks is at a new vertex and the other is at a visited one.

In the first case, either the walks move together, or the walker at the floor waits, while the second one goes down a step. But since this means before the step  $R$  was strictly below  $S$ , we get that after the downward move still  $z_R \leq z_S$ . In the second case, if  $R$  is the one at the new vertex, or if  $z_S > z_R$  then after the next step still  $z_R \leq z_S$ , so the only case we must worry about is that both walks are currently at the same vertex  $q$ , and  $q$  is a new vertex for  $S$ , while  $R$  has already visited it. To rule out this case, look at the first time  $R$  visited  $q$ . At that time, by the induction hypothesis, (items (1) and (3)),  $S$  was directly above  $R$ , and by the coupling rules,  $R$  would drop at least one step, and  $S$  would drop until it reached a vertex it has visited before (or the floor), prior to  $R$  making any sideways or upward move (and therefore prior to  $R$  returning to  $q$  and thus strictly before our current time). Thus when  $S$  reaches a non-new vertex, by the induction hypothesis (item (4)),  $q \in \text{Vis}_R \subset \text{Vis}_S$  contrary to our assumption. Thus this last case is dismissed and we have proven (3).

To see (4), roll back to the last induction step  $\sigma$  when  $S$  is not in a new vertex. By the induction hypothesis,  $\text{Vis}_S(\sigma)$  is downward closed and  $\text{Vis}_R(\sigma) \subset \text{Vis}_S(\sigma)$ . Seeing that  $\text{Vis}_S$  remains downward closed is obvious. To see  $\text{Vis}_R \subset \text{Vis}_S$  divide into cases according to which one moves. The only interesting case is when they make a simultaneous move in the  $x, y$  plane. However, if  $R(\sigma + 1) \notin \text{Vis}_S(\sigma)$  then so is  $S(\sigma + 1)$  (because  $\text{Vis}$  is downwards closed) and then  $S$  must drop until closing  $\text{Vis}(S)$  before  $\tau$ , so  $\text{Vis}_S$  contains the entire column  $[0, z_S(\sigma + 1)]$  above  $x_S, y_S$  which contains any points added to  $\text{Vis}(R)$  between  $\sigma + 1$  and  $\tau$ .  $\square$

To finish the proof of lemma ?? just take an instance of the coupling, and run it until  $R$  makes  $t$  moves. Since at each step of the coupling  $x_R = x_S, y_R = y_S$  and  $z_R \leq z_S$  we get that each time  $S$  hit a specific floor point,  $R$  hits it as well. The fact that  $z_R - \text{Wait}_R = z_S - \text{Wait}_S$  implies that in the remaining steps  $S$  has to complete  $t$  moves, he can at most reach the same  $z$  coordinate as  $R$  (if he goes straight down), so he cannot bypass the number of times  $R$  hits  $v$ .  $\square$

We now use the coupling lemma for the following useful corollary.

**Corollary 3.** For any floor point  $v$  and any  $t \in \mathbb{N}$ ,  $\mathbb{E}(V(t; v) | V(t; v) \neq 0) \leq \mathbb{E}V(t; \underline{0})$ .

*Proof.* Divide the probability space of all possible  $t$ -histories according to the path the walk takes until reaching  $v$  for the first time (Or not reaching it at all). Examine one path which does reach  $v$  in  $t$  steps, and let  $T_v$  denote the first time it hits  $v$ . Then, according to the coupling lemma, regardless of the history until time  $T_v$  ( $\text{Vis}(T_v)$ ), the expected number of times the walk will hit  $v$  in the *next*  $t$  steps is less or equal to the expected number of times a walk starting at  $v$  with no history (thus with no visited vertices) will visit  $v$  in  $t$  steps. But clearly there is no difference between the (expected) number of times a walk starting at  $v$  will hit  $v$  and the

number of times a walk starting at  $\underline{0}$  hits  $\underline{0}$ , so  $\mathbb{E}(V(t; v) | T_v, \text{Vis}(T_v)) \leq \mathbb{E}(\text{number of returns to } v \text{ in the next } t \text{ steps after } T_v | \text{Vis}(T_v)) \leq \mathbb{E}(V(t; \underline{0}))$ . Since this holds for any value of  $T_v$  and  $\text{Vis}(T_v)$ , the corollary follows.  $\square$

## 5. LOWER BOUND AND PROOF OF RECURRENCE

In the first few steps we will use the upper bound on the number of visits to a point to get a lower bound on the number of new vertices the walk visits, and consequently a lower bound on the number of times the walk hits the floor. Here we still do not need the coupling argument.

**Lemma 9.** *Denote by  $N(t)$  the number of different vertices the walk visits till time  $t$ , then  $\exists c > 0$  such that  $\mathbb{E}(N(t)) \geq ct/\sqrt{\log t}$ .*

*Proof.* Fix  $t$  and then

$$\mathbb{E}(N(t)) = \mathbb{E}(\#v : V(v) \neq 0) = \sum_v \mathbb{P}(V(v) \neq 0) =$$

by the definition of conditional expectation,

$$= \sum_v \frac{\mathbb{E}(V(v))}{\mathbb{E}(V(v) | V(v) \neq 0)} \geq$$

by corollary ??,

$$\begin{aligned} &\geq c \sum_v \frac{\mathbb{E}(V(v))}{\sqrt{\log t}} = \frac{c}{\sqrt{\log t}} \sum_v \mathbb{E}(V(v)) = \\ &= \frac{c}{\sqrt{\log t}} \mathbb{E}\left(\sum_v V(v)\right) = \frac{ct}{\sqrt{\log t}}. \end{aligned} \quad \square$$

We denote by  $DF(t)$  the number of *different floor* points the walk visits till time  $t$ . The next lemma bounds  $\mathbb{E}(DF(t))$ .

**Lemma 10.**  $\mathbb{E}(DF(t)) < Ct/\log t$

*Proof.* Since any two distinct floor points have different  $(x, y)$  coordinates, the number of different floor points visited by the ERW (till time  $t$ ), is bounded from above by the number of different points its  $(x, y)$  projection visits. But the projection of the ERW on the  $(x, y)$  plane is a simple random walk of length  $\leq t$ . Therefore  $\mathbb{E}(DF(t))$  is bounded above by the expected number of different vertices a SRW visits in  $t$  steps, which by Dvoretzky-Erdős [?] is  $< Ct/\log t$ .  $\square$

**Corollary 4.** There exists a  $c > 0$  such that  $\mathbb{E}(|\text{Vis}(t)|) \geq ct/\sqrt{\log t}$ .

*Proof.* Because  $|\text{Vis}(t)| = N(t) - DF(t)$ .  $\square$

Now we are ready to bound from below the expected number of times the ERW hits the floor.

**Lemma 11.** *Denote by  $F(t)$  the number of times the ERW hits the floor ( $z = 0$ ) until time  $t$ . Then there is a positive constant  $c > 0$ , independent of  $t$  such that  $\mathbb{E}(F(t)) \geq ct/\sqrt{\log t}$ .*

*Proof.* Look at the expected change of the  $z$  coordinate when the walk makes a single step. The walk has one of three behaviors according to the type of vertex it is currently in. If the walk is in a visited vertex — it acts as a SRW, so the expected change to the  $z$  coordinate is 0. If the walk is in a new vertex — it goes down a step — so the expected change to the  $z$  coordinate is  $-1$ , and finally, if the walk is on the floor then the expected change in the  $z$  coordinate is  $\frac{1}{5}$ . We get, by linearity of expectation,

$$\mathbb{E}(z(t)) = -1 \cdot \mathbb{E}(|\text{Vis}(t)|) + \frac{1}{5} \cdot \mathbb{E}(F(t)).$$

Since the walk always stays on the upper half space  $z \geq 0$ , we have  $\mathbb{E}(z(t)) \geq 0$  so  $\mathbb{E}(F(t)) \geq 5\mathbb{E}(|\text{Vis}(t)|) \geq 5ct/\sqrt{\log t}$  by the previous corollary.  $\square$

With the above estimates, all we need to do is combine our bounds on  $\mathbb{E}(F(t))$  and  $\mathbb{E}(DF(t))$  with corollary ?? to get:

**Theorem 2.** *There exists a constant  $c > 0$  such that  $\mathbb{E}(V(t; \underline{0})) \geq c\sqrt{\log t}$ .*

*Proof.* From linearity of expectation,

$$\mathbb{E}(F(t)) = \sum_{v \in \text{floor}} \mathbb{E}(V(v)) = \sum_{v \in \text{floor}} (\mathbb{E}(V(v) | V(v) \neq 0) \cdot \mathbb{P}(V(v) \neq 0)).$$

By corollary ??,  $\mathbb{E}(V(v) | V(v) \neq 0) \leq \mathbb{E}V(\underline{0})$  for any floor point  $v$ , so

$$\begin{aligned} \mathbb{E}(F(t)) &\leq \sum_{v \in \text{floor}} \mathbb{E}(V(\underline{0})) \cdot \mathbb{P}(V(v) > 0) = \mathbb{E}(V(\underline{0})) \cdot \sum_{v \in \text{floor}} \mathbb{P}(V(v) > 0) = \\ &= \mathbb{E}(V(\underline{0})) \cdot \mathbb{E}(DF(t)) \end{aligned}$$

so by lemmas ?? and ??,

$$\mathbb{E}(V(\underline{0})) \geq \frac{\mathbb{E}(F(t))}{\mathbb{E}(DF(t))} \geq \frac{ct/\sqrt{\log t}}{Ct/\log t} = c\sqrt{\log t}$$

For some absolute constant  $c > 0$ .  $\square$

**Theorem 3.** *ERW is recurrent.*

*Proof.* Assume to the contrary that there is a positive probability that an ERW  $R$  visits the origin exactly  $k$  times,  $k$  finite. Let  $\tau$  be the hitting time when  $R$  reaches the origin for the  $k$ -th time. Examining  $\text{Vis}_R(\tau)$  we see that there is a finite set  $X$  and a positive probability  $p_0 > 0$  such that if the walk  $R$  is currently at  $\underline{0}$  and  $\text{Vis}_R = X$  then  $\mathbb{P}(R \text{ will not return to } \underline{0}) \geq p_0$ . By the coupling lemma (lemma ??), if the walk  $R$  is at  $\underline{0}$ , and  $X \subset \text{Vis}(R)$ , then  $\mathbb{P}(R \text{ will not return again to } \underline{0}) \geq p_0$ . Since the set of visited vertices of a walk only increases, we conclude that once  $X \subset \text{Vis}_R$ , the number of times the walk returns to  $\underline{0}$  is dominated by a geometric random variable with parameter  $1 - p_0$ .

Next, notice that for any finite set of vertices  $Y$ , there is a positive probability  $p_1(Y)$  such that a walk currently at  $\underline{0}$  will visit all the vertices in  $Y$  before returning to  $\underline{0}$  with probability  $\geq p_1$ , regardless of the walk's history. (One such possible path is simply reaching each vertex  $v = (x, y, z) \in Y$  by walking on the floor [avoiding  $(0, 0)$ ] till you reach  $(x, y, 0)$ , then climbing slowly up until reaching the desired vertex, and finish the path by dropping on the  $(0, 0, *)$  column from high enough.) Thus the number of returns of ERW to  $\underline{0}$  before all vertices of  $X$  are visited is dominated by a geometric random variable with parameter  $1 - p_1(X)$ .

Combining the above we get that the total number of returns of ERW to  $\underline{0}$  is dominated by the sum of two geometric variables (with parameters  $1 - p_0, 1 - p_1$ ), and thus has finite expectation. But this contradicts theorem ??, (The expected number of returns till time  $t$  behaves like  $\sqrt{\log t}$ ), so ERW is recurrent.  $\square$

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