FILLED JULIA SETS WITH EMPTY INTERIOR ARE COMPUTABLE

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ABSTRACT. We show that if a polynomial filled Julia set has empty interior, then it is computable.

1. INTRODUCTION

Julia sets of rational mappings. We recall the main definitions of complex dynamics relevant to our results only briefly; a good general reference is the book of Milnor [Mil]. For a rational mapping R of degree $\deg R = d \geq 2$ considered as a dynamical system on the Riemann sphere

$$R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$$

the Julia set is defined as the complement of the set where the dynamics is Lyapunov-stable:

Definition 1.1. Denote F_R the set of points $z \in \hat{\mathbb{C}}$ having an open neighborhood $U(z)$ on which the family of iterates $R^n|_{U(z)}$ is equicontinuous. The set F_R is called the Fatou set of R and its complement $J_R = \hat{\mathbb{C}} \setminus F_R$ is the Julia set.

In the case when the rational mapping is a polynomial

$$P(z) = a_0 + a_1 z + \cdots + a_d z^d : \mathbb{C} \to \mathbb{C}$$

an equivalent way of defining the Julia set is as follows. Obviously, there exists a neighborhood of ∞ on $\hat{\mathbb{C}}$ on which the iterates of P uniformly converge to ∞. Denoting $A(\infty)$ the maximal such domain of attraction of ∞ we have $A(\infty) \subset F(R)$. We then have

$$J_P = \partial A(\infty).$$

The bounded set $\hat{\mathbb{C}} \setminus \operatorname{cl} A(\infty)$ is called *the filled Julia set*, and denoted K_P; it consists of points whose orbits under P remain bounded:

$$K_P = \{z \in \hat{\mathbb{C}} | \sup_n |P^n(z)| < \infty\}.$$

For future reference, let us list in a proposition below the main properties of Julia sets:

Proposition 1.1. *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function. Then the following properties hold:*

- J_R is a non-empty compact subset of $\hat{\mathbb{C}}$ which is completely invariant: $R^{-1}(J(R)) = J(R)$;
- $J_R = J_{R^n}$ for all $n \in \mathbb{N}$;
- J_R is perfect;
- if J_R has non-empty interior, then it is the whole of $\hat{\mathbb{C}}$;
- let $U \subset \hat{\mathbb{C}}$ be any open set with $U \cap J_R \neq \emptyset$. Then there exists $n \in \mathbb{N}$ such that $R^n(U) \supset J_R$;
- periodic orbits of R are dense in J_R.

Let us further comment on the last property. For a periodic point $z_0 = R^p(z_0)$ of period p its *multiplier* is the quantity $\lambda = \lambda(z_0) = DR^p(z_0)$. We may speak of the multiplier of a periodic cycle, as it is the same for all points in the cycle by the Chain Rule. In the case when $|\lambda| \neq 1$, the dynamics in a sufficiently small neighborhood of the cycle is governed by the Mean Value Theorem: when $|\lambda| < 1$, the cycle is *attracting* (*super-attracting* if $\lambda = 0$), if $|\lambda| > 1$ it is *repelling*. Both in the attracting and repelling cases, the dynamics can be locally linearized:

$$(1.1) \qquad\qquad\qquad \psi(R^p(z)) = \lambda \cdot \psi(z)$$

where ψ is a conformal mapping of a small neighborhood of z_0 to a disk around 0. By a classical result of Fatou, a rational mapping has at most finitely many non-repelling periodic orbits. The sharp bound on the number of such orbits is $2d - 2$; it is established by Shishikura. Therefore, we may refine the last statement of Proposition 1.1:

- *repelling periodic orbits are dense in J_R*.

In the case when $|\lambda| = 1$, so that $\lambda = e^{2\pi i\theta}$, $\theta \in \mathbb{R}$, the simplest to study is the *parabolic case* when $\theta = n/m \in \mathbb{Q}$, so λ is a root of unity. In this case R^p is not locally linearizable; it is not hard to see that $z_0 \in J_R$. In the complementary situation, two non-vacuous possibilities are considered: *Cremer case*, when R^p is not linearizable, and *Siegel case*, when it is. In the latter case, the linearizing map ψ from (1.1) conjugates the dynamics of R^p on a neighborhood $U(z_0)$ to the irrational rotation by angle θ (the *rotation angle*) on a disk around the origin. The maximal such *rotation domain* around z_0 is called a *Siegel disk*.

A different kind of a rotation domain may occur for a non-polynomial rational mapping R. A *Herman ring A* is a conformal image

$$\nu : \{z \in \mathbb{C} | \, 0 < r < |z| < 1\} \to A,$$

such that

$$R^p \circ \nu(z) = \nu(e^{2\pi i\theta}z),$$

for some $p \in \mathbb{N}$ and $\theta \in \mathbb{R} \setminus \mathbb{Q}$.

To conclude our discussion of the basic facts of rational dynamics, let us formulate the Fatou-Sullivan Classification of the types of connected components of the Fatou set of a rational mapping. The term *basin* in what follows will describe the set of points whose orbits converge to a given attracting or parabolic periodic orbit under the iteration of R.

Fatou-Sullivan Classification of Fatou Components . *For every connected component* $W \subset F_R$ *there exists* $m \in \mathbb{N}$ *such that the image* $H = R^m(W)$ *is periodic under the dynamics of* R*. Moreover, each periodic Fatou component* H *is of one of the following types:*

- *a component of the basin of an attracting or a super-attracting periodic orbit;*
- *a component of the basin of a parabolic periodic orbit;*
- *a Siegel disk;*
- *a Herman ring.*

Computability of subsets of $\mathbb{R}^n$**.** The computability notions that we apply for subsets of $\mathbb{R}^k$ belong to the framework of *computable analysis.* Their roots can be traced to the pioneering work of Banach and Mazur of 1937 (see [Maz]). The reader may find an extended exposition of the model of computation we are using in [BrC]. See [Wei] for a detailed discussion of the concepts of modern computable analysis. See also [BY] for a discussion of computability as applied to problems in Complex Dynamics.

Given a compact set $S \subset \mathbb{R}^k$, our goal is to be able to approximate the set S with an arbitrarily high precision. Here we ought to specify what do "approximate" and "precision" mean in this setting.

Denote $B(\bar{x}, r) \subset \mathbb{R}^k$ the closed Euclidean ball with radius r centered at $\bar{x}$. Denote $\mathcal{C}$ the set of finite unions $\cup B(\bar{x}_i, r_i) \subset \mathbb{R}^k$ of closed balls whose radii and coordinates of the centers are all dyadic rationals:

$$\mathcal{C} = \left\{ \bigcup_{i=1}^{k} B(\bar{x}_i, r_i) \ : \ \bar{x}_i \in \mathbb{Q}^k, \ r_i \in \mathbb{Q} \right\}.$$

For a set $K \subset \mathbb{R}^k$ denote its ε-neigborhood by $B(K, \varepsilon) = \cup_{\bar{x} \in K} B(\bar{x}, \varepsilon)$. Recall that the *Hausdorff metric* on compact subsets of $\mathbb{R}^k$ is defined by

$$d_H(X, Y) = \inf\{\varepsilon \ : \ X \subset B(Y, \varepsilon) \text{ and } Y \subset B(X, \varepsilon)\}.$$

That is, $d_H(X, Y)$ is the smallest quantity by which we have to "expand" X to cover Y and vice-versa.

In computable analysis a compact set $S \subset \mathbb{R}^k$ is called *computable*, if there exists a Turing Machine (TM) $M(m)$, such that on input m, $M(m)$ outputs an encoding of $C_m \in \mathcal{C}$ satisfying $d_H(S, C_m) < 2^{-m}$. That is, S is effectively approximable in the Hausdorff metric. This definition is quite robust. For example, it is equivalent to the computability of the distance function $d_S(x) = \inf\{|x-y| \ : \ y \in S\}$. In more practical terms, a set is computable if there exists a computer program which generates its image on a computer screen with one-pixel precision for an arbitrarily fine pixel size.

In the case of computing Julia sets, we are not actually computing a single set, but a set-valued function that takes the coefficients of the rational function $R(z)$ as an input, and outputs approximations of J_R. The machine computing J_R is given access to $R(z)$ through an *oracle* that can provide an approximation of any coefficient of $R(z)$ with an arbitrarily

high (but finite) requested precision. Such an oracle machine is denoted by M^ϕ. If P is the set of rational functions on which M^ϕ works correctly, then M^ϕ is said to be *uniform* on P. In case P is a singleton, the machine is said to be *non-uniform*. In other words, a uniform machine is designed to work on some family of inputs, whereas a non-uniform machine is only supposed to work on a specific input.

We remark that Julia sets have appeared in a somewhat similar context in the book of Blum, Cucker, Shub, and Smale [BCSS]. They considered the question whether a Julia set is decidable in the the Blum-Shub-Smale (BSS) model of real computation. The BSS model is very different from the computable analysis model we use, and can be very roughly described as based on computation with infinite-precision real arithmetic. Some discussion of the differences between the models may be found in [BrC] and the references therein. It turns out (see Chapter 2.4 of [BCSS]) that in the BSS model all, but the most trivial Julia sets are not decidable. More generally, sets with a fractional Hausdorff dimension, including the ones with very simple description, such as the Cantor set, are BSS-undecidable.

Statements of Results. The main purpose of the present note is to prove the following result:

Theorem 1.2. *For each $d \geq 2$ there exists an oracle Turing machine M^ϕ, with the oracle representing the coefficients of a polynomial of degree d with the following property. Suppose that $p(z)$ is a polynomial of degree d such that the filled Julia set K_p has no interior (so that $J_p = K_p$). Then K_p is computable by M^ϕ.*

The known results in this direction, which make the statement of the theorem interesting, are the following. Independently, the second author [Brv] and Rettinger [Ret] have demonstrated that *hyperbolic* Julia sets are polynomial-time computable by an oracle TM. In sharp contrast, in [BY] the second and third authors demonstrated the existence of non-computable Julia sets in the family of quadratic polynomials. The latter examples are given by polynomials with Siegel disks. It is shown in [BY] that there exist non-computable examples with rather wild topology. By a method of Buff and Chéritat [BC] the boundary of the Siegel disk can be made smooth, and as a consequence, the critical point is not in the boundary and the Julia set is not locally connected (see [Ro] for a discussion of the topological anomalies in such Julia sets).

As our present theorem shows, however, the notions of topological complexity and computational complexity are rather distinct. It covers, for example, the case of Cremer quadratics, which are also known to possess non-locally connected Julia sets. Further, we find the following statement plausible:

Conjecture. *The procedure of [BY] can be carried out so as to yield non-computable and locally connected Siegel quadratics.*

Our main result implies that the reason one is not able to produce high resolution computer-generated images of Cremer Julia sets is due to the enormous computational time required (whether such images would be very informative is another question). In view of this,

it is of great interest to obtain a classification of Julia sets into classes of computational hardness.

Finally, let us state a straightforward generalization of the main theorem to the case of a rational map without rotation domains (either Siegel disks or Herman rings). First note that by the Fatou-Sullivan Classification, every Fatou component of such a rational map belongs to the basin of a (super)attracting or parabolic orbit. A rational map f of degree d can have at most $2d-2$ of such orbits by Shishikura's result. Of these, the attracting orbits can be found algorithmically, with an arbitrary precision, as the condition $|(f^m)'(x)| < 1$ can be verified with a certainty by a TM. For an algorithm to ascertain the position of a parabolic orbit a finite amount of information is needed. For example, we can ask for its period m, and a good rational approximation to distinguish the point among the roots of $f^m(z) = z$.

Theorem 1.3. *Let f be a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ without rotation domains. Then its Julia set is computable in the spherical metric by an oracle Turing machine M^ϕ with the oracle representing the coefficients of f. The algorithm uses the nonuniform information on the number and positions of the parabolic orbits, as described above, as well as the (rational) values of the $\frac{1}{2\pi} \log$'s of the multipliers of the parabolic orbits.*

In particular, if there are no parabolic orbits the computation can be performed by a single oracle machine. As a function computed by an oracle Turing machine must be continuous on its domain (see e.g. [Brv]), we have the following corollary:

Corollary 1.4. *Denote $\mathcal{R}^d \subset \mathbb{C}^{2d-2}$ the set of parameters of degree d rational maps without rotation domains or parabolic orbits. Then a Julia set depends continuously, in Hausdorff sense on $\hat{\mathbb{C}}$, on the parameter in $\mathcal{R}^d$.*

While this is well-known (see the discussion in [Do]), it is notable that the proof of this dynamical statement is produced by a computer science argument. [1]

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2. Proof of the Main Theorem

Idea of the proof. The idea of the argument is to find two sequences of Hausdorff bounds for the filled Julia set. The *inside* ones are given by rational approximation of the periodic cycles of $p(z)$, accomplished by approximately solving a sequence of polynomial equations. The *outside* ones are given by preimages of a large disk centered at the origin. The algorithm stops when the difference between the two bounds is sufficiently small. This will always happen, provided the interior of the filled Julia set is empty.

We make no claims about the efficiency of our algorithm. Indeed, in most cases it will be far too slow to be practical. The existence of an efficient algorithm for the challenging cases, such as Julia sets with Cremer points, is an interesting open problem.

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[1] We owe this comment to J. Milnor

Lemma 2.1. *For every natural n we can compute a sequence of rationals $\{q_i\}$ such that*

$$(2.1) \qquad B(J_p, 2^{-(n+2)}) \Subset \bigcup_{i=1}^{\infty} B(q_i, 2^{-(n+1)}) \Subset B(J_p, 2^{-n}).$$

Proof. First of all, note that for any polynomial $Q(z)$, we can list rational approximations $r_1, r_2, \ldots, r_m$ of all the roots $\alpha_1, \alpha_2, \ldots, \alpha_m$ of $Q(z) - z$ with an arbitrarily good precision 2^{-k} (a classical reference is [Wey]).

Let $M > 0$ be some bound on $|Q''(z)|$ in the area of the roots. Then $|Q'(r_i)| > 1 + 2^{-k}M$ implies that $|Q'(\alpha_i)| > 1$, and in fact acts as a *certificate* that this is the case. It is easy to see that if, in fact, $|Q'(\alpha_i)| > 1$, then eventually we will find a rational approximation r_i of α_i for which we *know* that $|Q'(\alpha_i)| > 1$.

We now use the fact that the Julia set J_p is the closure of the repelling periodic orbits of p. Thus we can compute a sequence $\{q_i\}$ of rational approximations of all the repelling periodic points such that:

- For each i, there is a repelling periodic point α_i with $|q_i - \alpha_i| < 2^{-(n+3)}$, and
- for each repelling periodic point α, a $2^{-(n+3)}$-approximation of α will eventually appear in $\{q_i\}$.

We claim that (2.1) holds for the sequence $\{q_i\}$.

For any $x \in B(J_p, 2^{-(n+2)})$ there is a $y \in J_p$ with $|x - y| < 2^{-(n+2)}$. There is a repelling periodic point z with $|z - y| < 2^{-(n+3)}$, and the rational point q_i approximating it satisfies $|z - q_i| < 2^{-(n+3)}$, hence

$$|x - q_i| \le |x - y| + |y - z| + |z - q_i| < 2^{-(n+2)} + 2^{-(n+3)} + 2^{-(n+3)} = 2^{-(n+1)}.$$

So $x \in B(q_i, 2^{-(n+1)})$.

On the other hand, for any q_i there is a repelling $z \in J_p$ with $|q_i - z| < 2^{-(n+3)}$, hence $B(q_i, 2^{-(n+1)}) \subset B(J_p, 2^{-(n+1)} + 2^{-(n+3)}) \subset B(J_p, 2^{-n})$. $\qquad \square$

Proof of Theorem 1.2. Fix $m \in \mathbb{N}$. Our algorithm to find $C_m \in \mathcal{C}$ works as follows. Evaluate a large number $b > 0$ such that if $|z| > b$ then $|p(z)| > b$ and the orbit $p^n(z) \to \infty$. Set $D \equiv B(0, b)$. At the k-th step:

- compute the finite union $B_k = \cup_{i=1}^{k} B(q_i, 2^{-(m+1)}) \in \mathcal{C}$ as in the previous lemma;
- compute with precision $2^{-(m+3)}$ the preimage $p^{-k}(D)$, that is, find $D_k \in \mathcal{C}$ such that
$$d_H(\overline{D_k}, \overline{p^{-k}(D)}) < 2^{-(m+3)};$$
- if $D_k \subset B_k$ output $C_m = B_k$ and terminate. Otherwise, go to step $k + 1$.

We claim that this algorithm eventually terminates, and outputs an approximation $C_m \in \mathcal{C}$ with $d_H(C_m, K_p) < 2^{-m}$.

Termination. First, we note that $p^{-k}(D)$ is compactly contained in $p^{-(k-1)}(D)$ for all $k \in \mathbb{N}$. Since in our case $J_p = K_p$, we have

$$(2.2) \qquad K_p = \bigcap_{k \in \mathbb{N}} p^{-k}(D) = \overline{\{\text{repelling periodic orbits of } p\}}.$$

By the previous lemma, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have

$$B(K_p, 2^{-(m+2)}) \Subset B_k \Subset B(K_p, 2^{-m}).$$

By (2.2) there exists $k_1 \in \mathbb{N}$ such that for all $k \geq k_1$ we have

$$p^{-k}(D) \subset B(K_p, 2^{-(m+3)}), \text{ and hence } D_k \subset B(K_p, 2^{-(m+2)}).$$

Therefore, for $k \geq \max(k_0, k_1)$ we have $D_k \subset B_k$ and the algorithm will terminate.

Correctness. Now suppose that the algorithm terminates on step k. Since $D_k \subset B_k$ and $K_p \subset B(D_k, 2^{-(m+3)})$ we have $K_p \subset B(C_m, 2^{-(m+3)})$. On the other hand, $\cup\{q_i\} \subset K_p$, and thus $B_k = C_m \subset B(K_p, 2^{-(m+1)})$.

$\square$

Proof of Theorem 1.3. We will have to modify the way the set D_k is generated in the preceding algorithm. Note that the procedure which finds periodic points of f of period m, and verifies the conditions $|(f^m)'(x)| > 1$, or $|(f^m)'(x)| < 1$ with precision 2^{-m} will *eventually* certify every attracting periodic point. For each such point, we will also identify a Euclidean disk $U_x \in \mathcal{C}$ containing it with the property $f^m(U_x) \Subset U_x$.

Further, for each parabolic point x of period l let $U_x^k \in \mathcal{C}$ be a topological disk compactly contained in the immediate basin of attraction of x (i.e. the union of attracting petals), with the properties:

$$U_x^k \subset U_x^{k+1}, \text{ and } \cup_k U_x^k \text{ is an attracting petal.}$$

Such neighborhoods can easily be produced algorithmically given the non-uniform information on the parabolic orbits as stated in the Theorem. Our algorithm will then work as before, with the difference that the set D_k is defined as a $2^{-(m+3)}$-approximation of the *complement* of the preimage

$$W_k = f^{-k}(\bigcup_{x \text{ is attracting located so far}} U_x) \cup f^{-k}(\bigcup_{x \text{ is parabolic}} U_x^k).$$

By the Sullivan's Non-Wandering Theorem, we have $J_p = \cap(\hat{\mathbb{C}} \setminus W_k)$. The algorithm is verified exactly as in the previous theorem. $\square$

3. CONCLUSION.

We have shown that only rational maps with rotation domains (Siegel disks or Herman rings) may have uncomputable Julia sets. As was demonstrated earlier by the second and third authors [BY], such a possibility is indeed realized. To understand the cause of this phenomenon, consider the practical problem of plotting the Julia set of a Siegel quadratic $P_\theta(z) = z^2 + e^{2\pi i\theta}z$. The preimages of a large disk around the Julia set will rapidly converge to K_{P_θ}. However, when they are already quite close to J_{P_θ}, these preimages will still be quite far from the Siegel disk Δ_θ, as they will not fill the narrow fjords leading towards $\partial\Delta_\theta$. To have a faithful picture of a 2^{-n}-neighborhood of J_{P_θ}, one then needs to draw a

contour inside the Siegel disk, close to its boundary. In effect, this means knowing the inner radius of the Siegel disk. Indeed, we have the following easy statement:

Proposition 3.1. *Suppose the inner radius ρ_θ of Δ_θ is known. Then the Julia set J_{P_θ} is computable.*

Proof. The algorithm to produce the 2^{-n} approximation of the Julia set is the following. First, compute a large disk D around 0 with $P_\theta(D) \supsetneq D$. Then,

(I) compute a set $D_k \in \mathcal{C}$ which is a $2^{-(n+3)}$-approximation of the preimage $P_\theta^{-k}(D)$;

(II) set W_k to be the round disk with radius $\rho_\theta - 2^{-k}$ about the origin. Compute a set $B_k \in \mathcal{C}$ which is a $2^{-(n+3)}$-approximation of $P_\theta^{-k}(W_k)$;

(III) if D_k is contained in a $2^{-(n+1)}$-neighborhood of B_k, then output a $2^{-(n+1)}$-neighborhood of $D_k - B_k$, and stop. If not, go to step (I).

A proof of the validity of the algorithm follows along the same lines as that in Theorem 1.2. $\qquad\square$

The well-recognizable images of the "good" Siegel Julia sets, such as the one for $\theta_* = (\sqrt{5} - 1)/2$, the golden mean, are indeed computable by this recipe:

Proposition 3.2. *The inner radius ρ_{θ_*} is a computable real number.*

Proof. We appeal to the renormalization theory for golden-mean Siegel disks (see [McM]), which implies, in particular, that the boundary of Δ_{θ_*} is self-similar up to an exponentially small error. As a consequence, there exist $C > 0$, and $\lambda > 1$ such that

$$\inf\{|f^i(0)|, \; i = 0, \ldots, q_n\} < \rho_{\theta_*}(1 + C\lambda^{-n}),$$

where p_n/q_n is the n-th continued fraction convergent of θ_*. $\qquad\square$

In contrast, in the non-computable examples constructed in [BY] the boundary of the Siegel disk oscillates *unpredictably* at infinitely many scales. More precisely, recall the definition of the *conformal radius* $r(U, u)$ of a simply-connected domain U with a marked point u in its interior. For the unique holomorphic bijection Ψ from the unit disk $\mathbb{D}$ to U with $\Psi(0) = u$, and $\Psi'(0) > 0$ set

$$r(U, u) = \Psi'(0).$$

By the Koebe 1/4 Theorem of classical complex analysis, the inner radius of U as seen from u is at least $\frac{1}{4}r(U, u)$.

Now set $r_\theta = r(\Delta_\theta, 0)$. It is shown in [BY] that there exist parameter values $\tau \in \mathbb{R}/\mathbb{Z}$ for which the function $\theta \mapsto r_\theta$ is uncomputable on the set consisting of a single point $\{\tau\}$. On the other hand, we have:

Proposition 3.3. *Computability of J_{P_θ} implies computability of r_θ for Siegel parameter values θ.*

We note that a converse is also true, so computing the conformal radius is the only possible obstruction to drawing J_{P_θ}:
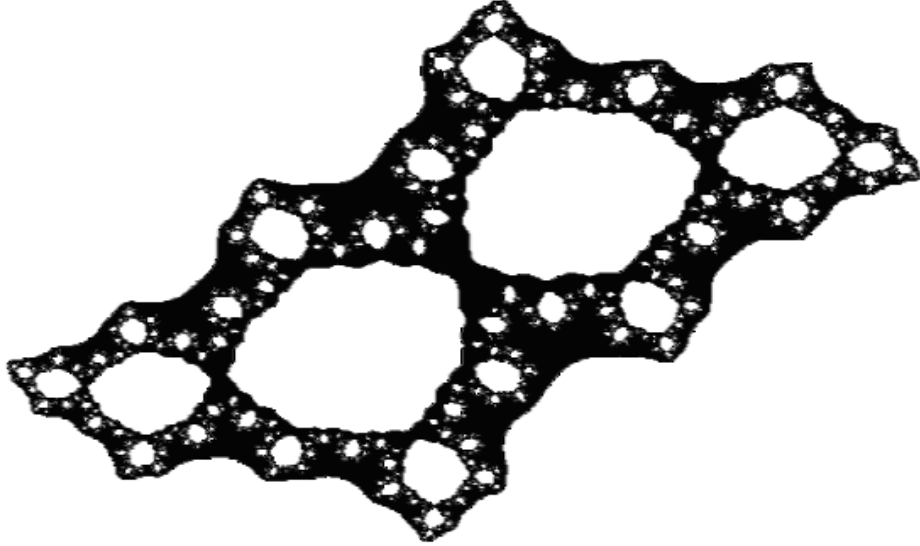
FIGURE 1. An image of the Siegel Julia set J_{θ_*} produced by the algorithm of Proposition 3.1

Proposition 3.4. *Suppose r_θ is computable, then so is J_{P_θ}.*

We will require the following facts of complex analysis:

Lemma 3.5. *Let (U, u) be as above, and suppose U_1 is a simply-connected subdomain of U containing the point u. Let $\epsilon > 0$ be such that $\partial U_1 \subset B(\partial U, \epsilon)$. Denote ρ and ρ_1 the inner radii of U and U_1 respectively, as seen from u. There exist explicit positive constants $C_1 = C_1(r(U, u))$, and $C_2 = C_2(r(U, u))$ such that*

$$C_1|\rho - \rho_1| \leq |r(U, u) - r(U_1, u)| \leq C_2\sqrt{\epsilon}.$$

The second inequality is an elementary consequence of Koebe Theorem and the constant may be chosen as $C_2 = 4\sqrt{r(U, u)}$ (see [RZ], for example). The other inequality may be found in many advanced texts on Complex Analysis, e.g. [Pom].

Proof of Proposition 3.4. We will show the computability of the inner radius of Δ_θ. The algorithm works as follows:

(I) For $k \in \mathbb{N}$ compute a set $D_k \in \mathcal{C}$ which is a 2^{-m}-approximation of the preimage $P_\theta^{-k}(D)$, for some sufficiently large disk D;

(II) evaluate the conformal radius $r(D_k, 0)$ with precision $2^{-(m+1)}$ (this can be done, for example, by using one of the numerous existing methods for computing the Riemann Mapping of a computable domain, see [Zho], for example), if necessairy, compute D_k or parts of D_k with a higher degree of precision;

(III) if $r(D_k, 0)$ is $2^{-m}C_1$-close to r_θ then compute the inner radius $\rho(D_k)$ around 0 with precision 2^{-m} and output this number. Else, increment k and return to step (I).

Termination. Let $K = K_{P_\theta}$ be the filled Julia set of P_θ. Then

$$\cap_{k=0}^\infty D_k = K \supset \Delta_\theta$$

and $D_0 \supset D_1 \supset D_2 \supset \dots$. Hence there will be a step k_0 after which

$$\text{dist}(\partial D_k, J_{P_\theta}) < \frac{2^{-2m}C_1^2}{C_2^2}.$$

$\partial \Delta_\theta \subset J_{P_\theta}$, and by Lemma 3.5 this implies that

$$|r(D_k, 0) - r(\Delta_\theta, 0)| = |r(D_k, 0) - r_\theta| < 2^{-m}\frac{C_1}{C_2}C_2 = 2^{-m}C_1,$$

and the algorithm will terminate on step (III).

Correctness. Now suppose the algorithm has terminated on step (III). $\Delta_\theta \subset D_k$, and Lemma 3.5 implies that

$$|\rho(D_k) - \rho_\theta| \leq 2^{-m}.$$

$\square$

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