Department of Mathematics
University of Toronto

APM421F / MAT1723F
Quantum Mechanics

Midterm Exam
October 21-29, 2009

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1. [5 points] Write out the expressions for
(a) Probability that in a state ψ the j^{th} component of momentum is in an interval $\Omega \subset \mathbb{R}$;
(b) The operator of angular momentum;
(c) Quantum Hamiltonian for the free particle;
(d) Quantum Hamiltonian for the Hydrogen atom.

2. [5 points] Which property of H is sufficient for conservation of probability for the Schrödinger equation $i\hbar\frac{\partial\psi_t}{\partial t} = H\psi_t,\ \psi_0 = \psi$? Prove it.

3. [5 points] Show that the following operators are symmetric
(i) $p = -i\frac{\partial}{\partial x}$;
(ii) $V:\ \psi(x) \to V(x)\psi(x)$, where $V(x)$ is a real and bounded function;
(iii) $i\frac{\partial^3}{\partial_x^3}$;
(iv) Laplacian;
(v) $H = -\frac{\hbar^2}{2m}\Delta + V(x)$ with $V(x)$ as in (ii).

4. [10 points] Let A be a bounded operator, H be a self-adjoint operator and ψ_t be the solution to $i\hbar\frac{\partial\psi_t}{\partial t} = H\psi_t$ and $\psi_0 = \psi$.
(a) Derive the equation
$$\frac{d}{dt}\langle\psi_t, A\psi_t\rangle = \langle\psi_t, \frac{i}{\hbar}[H, A]\psi_t\rangle.$$
(b) Prove that $e^{iHt/\hbar}Ae^{-iHt/\hbar}$ solves the equation $\partial_t A(t) = \frac{i}{\hbar}[H, A(t)]$.

5. [10 points] Show that the Laplacian operator, Δ, is self-adjoint.

6. [5 points] Give a formal proof of the following formulas:
   (a)
$$-i\hbar\widehat{\nabla_x\psi} = k\hat{\psi}(k);$$
   (b)
$$\widehat{x\psi} = i\hbar\nabla_k\hat{\psi}(k).$$

7. [5 points] Let $p = -i\hbar\frac{d}{dx}$. Find $i[p,x]$ $(dim = 1)$.

8. [20 points] Describe the spectra of the following operators
   (a)
$$p = -i\hbar\frac{d}{dx} \text{ (dimension} = 1),$$
   (b)
$$V: \ \psi(x) \to -\frac{1}{|x|+1}\psi(x),$$
   (c)
$$-\Delta,$$
   (d)
$$H = -\frac{\hbar^2}{2m}\Delta - 10|x|^3 + |x|^4,$$
   (e)
$$H = -\frac{\hbar^2}{2m}\Delta - (1+|x|)^{-2}.$$
   In the last case, describe possible location of isolated eigenvalues. Justify your answers.

9. [15 points] Estimate the ground state energy of $H = -\frac{\hbar^2}{2m}\Delta + |x|^4$ in three dimensions, using the variational principle with the test functions
$$\psi_\mu(x) = (8\pi)^{-1/2}\mu^{3/2}e^{-\mu|x|/2}.$$
   Is the estimate found lower or upper estimate of the true ground state energy?

10. [20 points] Below P_ψ stands for the orthogonal projection operator on a normalized wave-function ψ.

(a) Let H be a self-adjoint operator and ρ a bounded one. Prove that $\rho = e^{-\frac{iHt}{\hbar}}\rho_0 e^{\frac{iHt}{\hbar}}$ solves the equation

$$i\partial_t \rho = \frac{1}{\hbar}[H, \rho], \qquad \rho|_{t=0} = \rho_0.$$

(b) Let normalized vectors ψ_n evolve according to the Schrödinger equation, $i\hbar\frac{\partial\psi}{\partial t} = H\psi$. Show that the density matrix $\rho = \sum p_n P_{\psi_n}$ satisfies the equation

$$i\frac{\partial\rho}{\partial t} = \frac{1}{\hbar}[H, \rho] \; . \tag{1}$$

(c) Let ψ_i be eigenfunctions of H (i.e. $H\psi_i = \lambda_i\psi_i$). Show that $\rho = \sum_i p_i P_{\psi_i}$, for any $p_i \geq 0$, $\sum p_i = 1$, is a static solution of the equation $i\frac{\partial\rho}{\partial t} = \frac{1}{\hbar}[H, \rho]$.

Total 100 points