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DYNAMICAL SPECTRAL DETERMINATION AND RIGIDITY

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Abstract. The classical inverse problem (see [23]) asks to what extent it is possible to determine the shape of a domain $\Omega \subset \mathbb{R}^2$ (or of a manifold), by the knowledge of all eigenvalues of the Laplace operator on Ω (with e.g. Dirichlet boundary conditions). A dynamical version of this question can be stated by replacing the set of eigenvalues of Δ with the *Length spectrum*, that is the set of all lengths of all possible closed billiard orbits on Ω (or all closed geodesics in the case of manifolds). In these lectures, we will mention the deep connection between the Laplace inverse problem and the dynamical inverse problem; we will present in detail some results on the dynamical side (see [3, 12, 13]) and explore the possible outcomes of the current research in this direction.

Dedication

My interest in this topic resonated heavily with the research work of Steve Zelditch, whom I had the privilege to discuss with in several occasions (in front of a blackboard, on zoom, at various coffee shops, or again at a certain restaurant in Evanston). Steve passed away on September 2022, leaving a huge void in the community. In these lectures I will try to convey part of his monumental work, as seen from the dynamical side.

Lecture 1. An impressionistic picture of the problem

This section is meant to be a (somewhat) gentle introduction to the topic of (dynamical) spectral determination and rigidity.

The story that we are about to read about has very deep roots in Geometry, and describes a class of very important problems in PDEs (e.g. inverse problems), and –more recently– in Dynamics. The purpose of this section is to be a basin in which we collect results, and from which we will draw inspiration in different contexts in the following sections. All results that will be stated in this section will be mentioned with no proofs (or even hint to the proofs). Also, most of the basic concepts that will appear in this introduction will only be casually defined. Those ideas or tools that will be actually needed in the sequel will be given a proper treatment in the following lectures.

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Let M be a smooth closed (i.e. boundaryless and compact) manifold and let g be a smooth Riemannian metric on M. A metric g yields, in particular, the length $\|v\|_g$ of any vector $(x, v) \in TM$ in the tangent bundle. Given $p = (x, v) \in TM$, we let $\pi p = x$ be the projection on the manifold. Two Riemannian manifolds (M, g) and (M', g') are said to be *isometric* if there exists a diffeomorphism $f : M \to M'$ that carries the Riemannian structures back and forth, i.e. $f_* g' = g$.

Given a parametrization of a continuously differentiable curve $\gamma : I \to M$, where $I = [a, b] \subset \mathbb{R}$, g defines the length of γ by integrating the length of each tangent vector along the curve, that is:

$$L_g(\gamma) = \int_I \|\dot\gamma(s)\|_g ds.$$

A parametrization γ is said to be *proportional to arc-length* if $\|\dot\gamma\|_g$ is constant in I; it is said to be the *arc-length* parametrization if $\|\dot\gamma\|_g = 1$.

A curve γ is called a *geodesic with respect to the metric g* if it is parametrized proportionately to arc-length and for any $s \in I$ there exists a neighborhood $J \ni s$ so that $\gamma|_J$ is of minimal length among all curves that connect the boundary points $\partial\gamma|_J$.

1.1. Classical dynamics on (M, g) and the *Length spectrum*.

Geodesics satisfy the equation of parallel transport

$$\nabla_{\dot\gamma}\dot\gamma = 0,$$

where $\nabla_{\dot\gamma}$ is the covariant derivative (also induced by the Riemannian structure).[1] It follows (a standard result) that, given an element of the tangent bundle $(x, v) \in TM$, there exists a unique geodesic $\gamma : (-\varepsilon, \varepsilon)$ such that $\gamma(0) = x$, $\dot\gamma(0) = v$;

[1] You do not need to know what it means, but you can appreciate that this is a second order differential equation, so specifying an initial point and an initial direction guarantees a unique solution.

in particular $\|\dot{\gamma}\|$ is constant. Because of this homogeneity, it is customary to consider to the *unit tangent bundle*

$$T^1M = \{(x, v) \in TM \text{ s.t. } \|v\|_g = 1\}$$

given by those tangent vectors that are of unit length. By the standard Hopf–Ringo theorem, the geodesic γ admits a unique extension to $\mathbb{R}$ as a geodesic.

Each Riemannian metric thus induces a flow on the unit tangent bundle that we call the *Geodesic Flow*:

$$\begin{aligned} \Phi_g : \mathbb{R} \times T^1M &\to T^1M \\ (t, (x, v)) &\mapsto (x', v'), \end{aligned}$$

defined as follows: let $\gamma(s)$ be the arc-length parametrization of the unique geodesic passing by (x, v) at time 0 (i.e. $(x, v) = (\gamma(0), \dot{\gamma}(0))$); then we let $x' = \gamma(t)$ and $v' = \dot{\gamma}(t)$. Since solutions to ODEs depend smoothly on initial conditions, the Geodesic Flow is smooth.

A point $p = (x, v) \in T^1M$ is called *periodic* if there exists $T > 0$ so that $\Phi_g(t, p) = p$. Such a T is called a *period* of p: notice that if T is a period of p, then so is any $T' \in T \cdot \mathbb{Z}$. Given a periodic point p, we call *prime period* of p the smallest $T > 0$ in $\mathbb{R}$ which is a period of p. Observe that the curve $\gamma : [0, T] \to M$ given by $\gamma(t) = \pi\Phi_g(t, p)$ is a closed curve of length T. By construction, it is a geodesic.

The main character of these notes is the following object:

$$\mathrm{LS}(M, g) = \{T > 0 \text{ s.t. } T \text{ is a period of some periodic point}\} \subset \mathbb{R}_{>0}.$$

We call the above set the *Length Set*[2] of (M, g). Notice that it is an "unformatted" set of numbers. Refinements of this concept will appear below, in which we consider multiplicities, markings, etc., but for now we are only considering the set as a subset of $\mathbb{R}$.

1.2. Quantum dynamics on (M, g) and the *Laplace spectrum*.

Periodic orbits can be regarded as "stable" motions of the geodesic flow, which is –physically– the classical free motion of a particle on the manifold. The *quantum* free motion of a particle on the manifold is, on the other hand, determined by a PDE, which is the free Schrödinger equation

$$i\hbar\dot{\Psi} = \Delta_g \Psi,$$

where Δ_g is the Laplace–Beltrami operator associated[3] to the metric g. Clearly, eigenfunctions of Δ_g correspond to "stable" quantum states; the corresponding

[2]The reason we do not call it *Length Spectrum* at this stage is that a *spectrum* is implicitly assumed to contain information about multiplicities, and we cannot do this in this generality.

[3]In case you *really* want to know, it is defined, in coordinates, as

$$\Delta_g = \frac{1}{\sqrt{|g|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} g^{ij} \sqrt{|g|} \frac{\partial}{\partial x_j}$$

where $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$, g^{ij} is the inverse matrix and $|g| = \det g_{ij}$.

eigenvalues are –physically– the energy levels of the state. The spectrum of the Laplacian is conventionally defined as[4]:

$$\Delta\mathrm{S}(M,g) = \{\lambda \in \mathbb{R}_{\geq 0} \text{ s.t. } \exists \Psi \in L^2(M,g) \text{ with } \Delta\Psi + \lambda^2\Psi = 0\}.$$

Since M is compact, $\Delta\mathrm{S}(M,g)$ is discrete; typically one considers $\Delta\mathrm{S}$ to be a *set with multiplicities* in case some of the eigenvalues are degenerate.

1.3. **The Wave Trace formula.**

On top of the philosophical affinity, there is a strong, surprising and fascinating mathematical connection between $\mathrm{LS}(M,g)$ and $\Delta\mathrm{S}(M,g)$, which can be described using the *Wave Trace Formula*. The *wave trace* is defined as the trace of the operator

$$U(t) = e^{it\sqrt{-\Delta_g}} : L^2(M,g) \to L^2(M,g)$$

and can be defined as

$$S(t) = \sum_{\lambda \in \Delta\mathrm{S}(M,g)} e^{i\lambda t}$$

(the convergence of the sum on the right hand side is intended in the sense of distributions). The following amazing result holds

Theorem 1.1 (Poisson's relation for closed manifolds [9, 8, 15])**.** *If $t \notin \pm\mathrm{LS}(M,g) \cup \{0\}$, the distribution S is C^∞ at t.*

More specifically, one can in fact write a singularity expansion:

$$S(t) = e_0(t) + \sum_{L \in \Delta\mathrm{S}(M,g)} e_L(t) \mod C^\infty,$$

where e_0 and e_L are distributions with singularities at just one point (resp. $t = 0$ and $t = L$). If there are no two distinct (modulo time-reversal) geodesics with the same length (this is a generic feature of g), then we can indeed conclude that $S(t)$ is singular at t if and only if $t \notin \pm\mathrm{LS}(M,g) \cup \{0\}$, that is to say that *generically, the Laplace spectrum determines the Length spectrum.*

Remark 1.2. The Poisson's relation also holds for manifolds with boundary thanks to the work of Andersson–Melrose [1] and Guillemin–Melrose [21]. See [28] for an in-depth exposition of the result.

1.4. **Spectral determination.**

It follows by the definitions that any two isometric manifolds will have the same Length Set and the same Laplace Spectrum. The *Laplace inverse problem*, or the (quantum) problem of spectral determination can be stated as follows:

> If two manifold (M,g) and (M',g') have the same Laplace spectrum, are they necessarily isometric?

Corresponding questions can be formulated for the Length set $\mathrm{LS}(M,g)$:

> If two manifolds (M,g) and (M',g') have the same Length Set , are they necessarily isometric?

[4]Below, the notation $L^2(M,g)$ is shorthand for $L^2(M,\mathrm{Vol}_g)$, where Vol_g denotes the Borel measure on M induced by the Riemannian Volume associated to g.

It was first realized by Milnor that, in general, both these problems has a negative answer. Milnor found a (somewhat exotic) counterexample in 1964 (two non-isometric but isospectral flat structures on $\mathbb{T}^{16}$), but there are famous counterexamples even in constant negative curvature introduced in the late 1970's by Marie-France Vigneràs [31] and later developed by T. Sunada [30]. Notice that, in view of the Poisson relation mentioned in the previous section, a negative answer to the quantum question implies a negative answer to the classical question.

A positive answer to the problem of Laplace determination for surfaces was provided, under symmetry and analiticity assumptions, by Steve Zelditch in the late 1990's:

Theorem 1.3 (Zelditch [32]). *Assume that (M, g) and (M, g') are real* analytic *simple* surfaces of revolution *for which the meridian geodesic length is simple. Then $\Delta S(M, g) = \Delta S(M, g')$ implies $g = g'$.*

Remark 1.4. The above result hinges on three crucial assumption: *symmetry*, *analiticity* and some kind of generic *non-degeneracy*. These assumptions are typical in every result about determination, unless additional properties are added, as we will discuss in the next subsection.

An extremely interesting, in-depth and (unlike this section) rigorous piece on the topic of Laplace determination is the fantastic survey [34] by Zelditch.

1.5. The Marked Length Spectrum.

It becomes then clear that additional information is needed to state an interesting problem, and –as it turns out, in order to obtain this additional information it is necessary to add some additional structure.

Brilliant affirmative results are in fact available; most of them pertain the class of *negatively curved manifolds*.

An important feature (for us) of such manifolds is that each *free homotopy class* has a *unique geodesic representative*. This is to say that any closed geodesic is unique among its free homotopy class. This allows to add some information (a *marking*) to the length spectrum, and this extra information is crucial in the problem of spectral determination.

Let M be a smooth closed manifold and let $\mathcal{C}(M)$ denote the set of *free homotopy classes* of M; in other word, each $c \in \mathcal{C}(M)$ is a closed loop in M modulo homotopy.

Proposition 1.5. *Let M be a smooth closed manifold, and g a smooth Riemannian metric on M of negative curvature. Then, for any $c \in \mathcal{C}(M)$, there exists a* unique *representative $\gamma_c \in c$ that is a unit-speed geodesic with respect to g.*

Leveraging on the above mentioned uniqueness property, we can define the *Marked Length Spectrum* of a negatively curved Riemannian manifold.

Definition 1.6. Let (M, g) be a negatively curved Riemannian manifold, then we can define $\mathrm{MLS}_{M,g} : \mathcal{C}(M) \to \mathbb{R}_{\geq 0}$ as the map:

$$c \mapsto \mathrm{len}_g(\gamma_c)$$

The length spectrum $\mathrm{LS}(M, g)$ is just the image of MLS; in this sense, the map MLS defines a *marking* of the length spectrum, as it specifies which orbit(s) correspond to a given length. In absence of such a marking, it seems (for now) that the information contained in the length spectrum (or in the Laplace spectrum) is insufficient to fully recover the metric

Adding this extra bit of *topological* information turns out to be crucial; let us reformulate the *spectral determination* question as follows:

Question. Assume that (M, g) and (M, g') are negatively curved manifolds that have the same Marked Length Spectrum, are they isometric?

The first results in this direction gave a positive answer to a question that involves a weaker notion of determination. In order to discuss such notions, we need to introduce the notion of deformations of a metric.

Definition 1.7. Let M be a smooth manifold and $(g_t)_{t\in(-\varepsilon,\varepsilon)}$ a smooth family of smooth Riemannian metrics. The family g_t is said to be a *deformation* of g_0. A family is said to be *trivial* if there exists a family of diffeomorphisms $f_t : M \to M$ so that $g_t = f_{t*}g_0$. A family is said to be isospectral if $\mathrm{LS}_{M,g_t} = \mathrm{LS}_{M,g_0}$. Clearly, any trivial family is isospectral.

Remark 1.8. It will be proved later (at least in a special case, but the reason behind the result is the same) that deformations automatically preserve markings. Hence, any isospectral family automatically preserves the MLS, where it is defined.

We now introduce the notion of *spectral rigidity*

Definition 1.9. Let M be a smooth manifold and g a Riemannian metric on M. g is said to be *spectrally rigid* if it admits no non-trivial isospectral deformations.

The first results about rigidity appeared in the 1970's for surfaces.

Theorem 1.10 (Guillemin, Kazhdan [20])**.** *Let M be a closed manifold of dimension* 2 *and g be a negatively curved Riemannian metric; then g is spectrally rigid.*

The above result was generalized to arbitrary dimension in the 1990's

Theorem 1.11 (Croke, Sharafutdinov [11])**.** *Let M be a closed manifold and g be a negatively curved Riemannian metric; then g is spectrally rigid.*

The condition of negative curvature can be slightly relaxed, but not to an arbitrary extent; for instance C. Gordon and Wilson proved in [17] the existence of 1-parameter families of isospectral Riemannian manifolds; such manifolds are algebraic in nature and they are not "non-positively curved".

We now proceed to introduce the proper notion of *spectral determination*

Definition 1.12. Let M be a smooth manifold, $\mathcal{G}(M)$ be a certain class of Riemannian metrics on M. Then $g \in \mathcal{G}(M)$ is said to be *spectrally determined in* $\mathcal{G}(M)$ if the following holds: if $g' \in \mathcal{G}(M)$ is so that $\mathrm{MLS}_{M,g} = \mathrm{MLS}_{M,g'}$, then there exists a diffeomorphisms $f : M \to M$ so that $g' = f_*g$ (i.e. g and g' are isometric).

The first notable result concerning spectral determination appeared in 1990, and it is an independent result of Otal and Croke about negatively curved surfaces.

Theorem 1.13 (Otal [27] $\perp$ Croke [10])**.** *Let M be a* 2*-dimensional manifold and g a negatively curved metric. Then g is spectrally determined among all negatively curved metrics.*

The result was recently (and quite spectacularly!) generalized to arbitrary dimension by Guillarmou and Lefeuvre,

Theorem 1.14 (Guillarmou, Lefeuvre (and Knieper) [19, 18])**.**
Let M be a n-dimensional manifold and g a negatively curved metric. Then g is spectrally determined among all negatively curved metrics in a neighborhood of g.

Remark 1.15. Both above results are actually true in a slightly larger class of metrics that are called *Anosov* metrics: a metric g is called *Anosov* if it a non-positive curvature metric so that the associated geodesic flow Φ_g is an Anosov flow.

Remark 1.16. The result by Guillarmou–Lefeuvre can be referred to as *local determination* and it is a property of g which is stronger than rigidity but weaker than determination. The fact that every Anosov metric is *locally determined* can be rephrased as "isospectral Anosov metrics form a discrete set" (in the natural topology). Local determination is quite stronger than spectral rigidity: for instance there exist sequences g_n of metrics that converge to g, but that cannot sit on any deformation of g.

1.6. Global, local or relative markings.

In more general settings we do not have the luxury of Proposition 1.5, and it is not clear even how to provide a marking. In other situations one might not want to use the full marking and it is interesting to explore what can be recovered with only partial information.

A weak form of marking is to endow the Length Set with multiplicities associated to each length (i.e. the number of distinct geodesics that share the same length). This is well defined for systems where the number of geodesics of length bounded by a certain constant is discrete, but fails miserably otherwise (think about the sphere S^2, or the torus $\mathbb{T}^2$ where geodesics come in families)

One could explore a "local" form of marking, where we only consider geodesics traversing some neighborhood of T^1M. This could be useful if additional rigidity properties are assumed (e.g. analiticity) or if only a "local" statement about isometricity (*is that a word?*) is desired.

One could also explore a "relative" form of marking, in which a global topological model is absent, but we have a homeomorphism between the "orbit spaces" of $(M.g)$ and (M, g') so that we can identify which orbit of $\Phi_{g'}$ corresponds to which orbit of Φ_g (any two such flows are called *orbit equivalent*). A question in this sense could be: which properties do g and g' have to satisfy for the following to hold: g and g' are isometric if and only if the length of corresponding orbits agree.

1.7. And now... billiards!

The content of our lectures will focus on stating corresponding questions and giving some partial answers in the setting of *Dynamical billiards*. A cheap definition of billiard dynamics is *geodesic flow on a flat surface with a (smooth enough) boundary*, but a number of very specific problems arise in this situation. The quantum spectral determination problem in the case of billiards[5] has been famously paraphrased (see [23]) by M. Kac in 1967 as: *Can you*

[5] In order for the question to be well-posed one needs to fix boundary conditions; a standard choice is e.g. , Dirichet boundary conditions.

hear the shape of a drum?. Using techniques similar to the ones mentioned earlier in the work of Sunada, in 1992 C. Gordon, Webb and Wolpert proved (see [16]) the existence of domains (bounded by a piecewise-linear curve) that are isospectral but not isometric. The question for smooth domains is still open in general. A breakthrough result is the following theorem of Zelditch, published in 2008.

Theorem 1.17 (Zelditch [33])**.** *Let Ω and Ω' be chosen among a generic class of analytic domains which are axially-symmetric. Then if $\Delta S(\Omega) = \Delta S(\Omega')$, the domains Ω and Ω' are isometric.*

The most recent available result on the Laplace problem is due to Hezari and Zelditch:

Theorem 1.18 (Hezari, Zelditch [22])**.** *If Ω is a domain bounded by a C^∞ curve and its Laplace spectrum is the same as the spectrum of a domain bounded by an ellipse of sufficiently small eccentricity, then Ω is such an ellipse.*

The standing conjecture in this field has been formulated by Sarnak in 1990:

Conjecture (Sarnak [26])**.** *Every domain bounded by a C^∞ curve is* locally *determined by its Laplace Spectrum.*

The conjecture is still open today. There was basically no progress whatsoever in the category of smooth domains for over 25 years. Now, however, we are in far better shape than we were 10 years ago.

The story we are about to tell describes what happened in these 10 years.

Notations used throughout the lectures. For $\ell > 0$, we denote with $\mathbb{T}_\ell = \mathbb{R}/\ell\mathbb{Z}$ the one-dimensional torus of circumference ℓ; we also identify $\mathbb{S}^1 = \mathbb{T}_{2\pi}$.

With $C^r(X)$ we denote the space of functions from X to $\mathbb{R}$ that are differentiable r times and with continuous r-th derivative. We denote with C^∞ the space of infinitely differentiable functions and with C^ω the space of real analytic functions. X is assumed to have enough structure for the definitions to make sense (e.g. C^r-manifold, smooth manifold, analytic manifold)

Lecture 2. Introduction of the main characters

We begin with the definition of the class of *strongly convex domains*.

Definition 2.1. Let $r \geq 2$ (or $r = \infty$ or $r = \omega$) and $\ell > 0$; let $\gamma : \mathbb{T}_\ell \to \mathbb{R}^2$ be an arc-length parametrization of a C^r Jordan curve; assume furthermore that its curvature nowhere vanishes (i.e. $|\dot\gamma(s)| = 1$ and $\ddot\gamma(s) \neq \mathbf{0}$ for any $s \in \mathbb{T}_\ell$).

The curve γ identifies a bounded domain $\Omega \subset \mathbb{R}^2$ which we call a C^r *strongly convex planar domain*. We denote the set of all C^r strongly convex planar domains by $\mathcal{M}^r$.

Remark 2.2. A strongly convex domain Ω is, in particular, strictly convex, but there exist smooth strictly convex domains that are not strongly convex (for instance, an oval with a "flat spot" where the curvature vanishes).

Exercise 2.3. Show that for any $\theta \in \mathbb{S}^1$ there exists a unique $s \in \mathbb{T}_\ell$ so that $\dot\gamma(s)$ forms an angle θ with the positive horizontal semi-axis. Show that the map $\theta : \mathbb{T}_\ell \to \mathbb{S}^1$ is a C^{r-1} diffeomorphism (Hint: show that $\theta'(s) = |\ddot\gamma(s)|$ for any $s \in \mathbb{T}_\ell$).

Using the above parametrization, the space $\mathcal{M}^r$ can be equipped with a natural metric: each domain Ω can be uniquely represented as a C^{r-1}-function $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ parametrizing its boundary. We can thus define naturally:

$$d(\Omega_0, \Omega_1) = \|\gamma_0 - \gamma_1\|_{C^{r-1}}$$

2.1. Birkhoff billiards.

Billiards inside convex domains were first introduced in the 1920's by George David Birkhoff.[6] Birkhoff introduced such systems in [6].

Let $\Omega \in \mathcal{M}^r$, with $r \geq 3$: we call Ω the *Billiard Table*. Let $X_\Omega = \partial\Omega \times [0, \pi]$ (consider it as a C^r manifold with boundary): we call X_Ω the *phase space of the billiard in* Ω. For any $z = (x, \psi) \in X_\Omega$, consider the oriented line $\Lambda(x, \psi)$ passing through $x \in \partial\Omega$ and forming an angle ψ (measured counterclockwise) with the positively oriented tangent vector to $\partial\Omega$ at x. We let x' denote the unique[7] other point of intersection of $\Lambda(x, \psi)$ with $\partial\Omega$ and ψ' the angle (measured clockwise) between the line and the positively oriented tangent vector to $\partial\Omega$ at x' (if $\psi = 0$ or $\psi = \pi$, we let $x' = x$ and $\psi' = \psi$).

If we imagine that a point particle is emitted from x in the direction identified by the angle ψ, then its trajectory will follow along the line $\Lambda(x, \psi)$ until the particle hits $\partial\Omega$; then it will undergo elastic reflection and its trajectory would follow the line $\Lambda(x', \psi')$ until the next collision.

Definition 2.4. The map $T_\Omega : X_\Omega \to X_\Omega$ given by $T_\Omega(x, \psi) \mapsto (x', \psi')$ is called the *Billiard map* associated to Ω.

Exercise 2.5. Show that $\Lambda(x, \psi)$ and $\Lambda(x', \pi - \psi')$ are the same line with opposite orientation. Let $\mathcal{I} : X_\Omega \to X_\Omega$ be the idempotent map $\mathcal{I} : (x, \psi) \mapsto (x, \pi - \psi)$; then show that $T_\Omega \circ I = I \circ T_\Omega^{-1}$. We call $\mathcal{I}$ the *involution map*.

Remark 2.6. Notice that we are asking $r \geq 3$ in the definition of a billiard table. The reason why we need some extra smoothness is to guarantee certain basic properties of the Billiard dynamics. For example, it is possible to construct a C^2

[6] It is the same *Birkhoff* as the *Birkhoff Ergodic Theorem*

[7] Uniqueness follows from convexity of Ω

domain Ω with the property that there is an orbit that accumulates on a point $(\bar{x}, 0) \in X_\Omega$. In particular, on such a domain, there exists a billiard trajectory of finite length, i.e. the *billiard flow would be incomplete.*

Exercise 2.7. It is a nice exercise to construct an example of a domain with the properties described in Remark 2.6: you really just need to construct a small arc of $\partial\Omega$ around the accumulation point…

For the remainder of this part, we fix some Ω and drop it from our notation. Let $\ell = |\partial\Omega|$ the perimeter of Ω and let us fix γ to be an arc-length parametrization of $\partial\Omega$.[8] We let $\mathcal{X}_\ell = \mathbb{T}_\ell \times [0, \pi]$; then $\gamma \times \mathrm{Id}$ is a C^r diffeomorphism from $\mathcal{X}_\ell$ to X_Ω. We denote with $\mathcal{T}$ the billiard map in these coordinates $\mathcal{T} : \mathcal{X}_\ell \to \mathcal{X}_\ell$, by $\hat{\mathcal{X}}_\ell = \mathbb{R} \times [0, \pi]$ a lift of $\mathcal{X}_\ell$ and by $\hat{\mathcal{T}} : \hat{\mathcal{X}}_\ell \to \hat{\mathcal{X}}_\ell$ the lift of $\mathcal{T}$ which fixes $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{\pi\}$ (i.e. so that $\hat{\mathcal{T}}(s, 0) = (s, 0)$ and $\hat{\mathcal{T}}(s, \pi) = (s, \pi)$ for any $s \in \mathbb{R}$).

2.2. Generating function.

Let $L : \partial\Omega \times \partial\Omega \to \mathbb{R}$ be the continuous map:

$$L(x, x') = \mathrm{dist}(x, x')$$

where $\mathrm{dist}(\cdot, \cdot)$ denotes the standard Euclidean distance in $\mathbb{R}^2$. We also define $\mathcal{L} : \mathbb{T}_\ell \times \mathbb{T}_\ell \to \mathbb{R}$ as $\mathcal{L}(s, s') = L(\gamma(s), \gamma(s'))$; notice that $\mathcal{L}$ is as smooth as $\partial\Omega$ except on the diagonal $\Delta = \{(s, s) \text{ s.t. } s \in \mathbb{T}_\ell\} \subset \mathcal{X}_\ell \times \mathcal{X}_\ell$.

Exercise 2.8. Show that

$$\partial_1 \mathcal{L}(s, s') = -\cos(\psi) \qquad\qquad \partial_2 \mathcal{L}(s, s') = \cos(\psi')$$

where ∂_i denotes the partial derivative with respect to the i-th component and ψ (resp. ψ') is the angle that the segment $\overline{\gamma(s)\gamma(s')}$ makes with the positively oriented tangent vector to $\partial\Omega$ at the point identified by s (resp. s').

Exercise 2.9. Show moreover that

$$\partial_{12} \mathcal{L}(s, s') < 0$$

It is customary to introduce coordinates (s, r) on $\mathcal{X}_\ell$, where $r = -\cos(\psi) \in [-1, 1]$; the function[9] $-\mathcal{L}$ is called the *generating function* of the billiard map $\mathcal{T}$ in these coordinates[10], in the sense that

$$\mathcal{T}(s, r) = (s', r') \iff \begin{cases} r &= -\partial_1 \mathcal{L}(s, s') \\ r' &= \partial_2 \mathcal{L}(s, s') \end{cases}$$

An immediate consequence of the existence of a generating function is the following result:

Lemma 2.10. *The map $\mathcal{T}$ is area-preserving in (s, r) coordinates.*

Proof. Compute the differential of $\mathcal{L}$:

$$d\mathcal{L} = -r ds + r' ds'$$

[8] To fix ideas we can choose it so that $\dot{\gamma}(0)$ is the unit vector $(1, 0)$ (but this choice is rather arbitrary).

[9] The choice of sign is –arguably– purely conventional, but with *this* choice of sign, the generating function has the same features as the *physical action*.

[10] Such maps are called *twist maps*

This implies that the 1-form $\alpha = rds$ is $\mathcal{T}$-invariant modulo total differentials, i.e. $\mathcal{T}_*\alpha = \alpha' - d\mathcal{L}$. By differentiating once again this expression we conclude that the 2-form $d\alpha = dr \wedge ds$ is $\mathcal{T}$-invariant, which concludes the proof.

Alternatively, we can just perform the simple computation of the Jacobian matrix

$$\frac{\partial s'}{\partial s} = -[\partial_{12}\mathcal{L}]^{-1}\partial_{11}\mathcal{L} \qquad \frac{\partial s'}{\partial r} = -[\partial_{12}\mathcal{L}]^{-1}$$
$$\frac{\partial r'}{\partial s} = \partial_{12}\mathcal{L} - [\partial_{12}\mathcal{L}]^{-1}\partial_{11}\mathcal{L}\cdot\partial_{22}\mathcal{L} \qquad \frac{\partial r'}{\partial r} = -[\partial_{12}\mathcal{L}]^{-1}\partial_{22}\mathcal{L},$$

which immediately implies that the determinant is 1. □

In (s, ψ) coordinates the invariant area form is written as

$$d\text{Liouv} = \sin\psi ds \wedge d\psi \tag{2.1}$$

it is also called the *Liouville measure* of the billiard map

Generating functions are particularly useful to show existence of *periodic points* for the billiard map.

Definition 2.11. A point $z = (x, \psi) \in X_\Omega$ is said to be q-periodic if $T^q z = z$; notice that if z is q-periodic, then it is also kq-periodic for any $k \in \mathbb{N}$. We call *prime period of* z the smallest q so that $T^q z = z$. If $z \in \mathcal{X}_\ell$ is periodic of prime period q, let $\tilde{z} \in \mathbb{R} \times [0, \pi]$ be a lift of z; then $\hat{\mathcal{T}}\tilde{z} = \tilde{z} + (p\ell, 0)$ for some $p \in \mathbb{Z}$; we call p the *winding number* of z and the ratio $\omega = p/q$ the *rotation number of* z.

Remark 2.12. The rotation number is a topological invariant of a periodic orbit, in the sense that if two billiard maps are topologically conjugate, then a periodic orbit of rotation number ω is mapped by the conjugacy to a periodic orbit of the same rotation number.

For any $q > 1$, let $L_q : \partial\Omega^q \to \mathbb{R}$ defined as:

$$L_q(x_0, \cdots, x_{q-1}) = \sum_{j=0}^{q-1} L(x_j, x_{j+1 \mod q}).$$

The function L_q maps an ordered q-tuple of points on $\partial\Omega$ to the perimeter of the polygon inscribed in Ω that is obtained by connecting the points respecting their order in the q-tuple. We also define $\mathcal{L}_q : \mathbb{T}^q_\ell \to \mathbb{R}$ given by:

$$\mathcal{L}_q(s_0, \cdots, s_{q-1}) = L_q(\gamma(s_0), \cdots, \gamma(s_{q-1})).$$

Definition 2.13. For any $q \in \mathbb{Z}_{\geq 2}$ define the open set

$$\hat{\mathbb{T}}^q_\ell = \{(s_0, \cdots, s_{q-1}) \in \mathbb{T}^q_\ell \text{ s.t. } s_j \neq s_{j+1 \mod q} \text{ for any } j\}$$

and correspondingly:

$$b\Omega^q = \{(\gamma(s_0), \cdots, \gamma(s_{q-1})) \text{ s.t. } (s_0, \cdots, s_{q-1}) \in \hat{\mathbb{T}}^q_\ell\} \subset \partial\Omega^q.$$

Observe that $\mathcal{L}_q$ is smooth on $\hat{\mathbb{T}}^q_\ell$; observe moreover that if a billiard orbit $\mathbf{s} \in \hat{\mathbb{T}}^q$, then it is necessarily the trivial orbit i.e. $(s_0, \cdots, s_{q-1})$ is so that $s_0 = s_1 = \cdots = s_{q-1}$.

Exercise 2.14. Let $z \in \mathcal{X}_\ell$ and define $z_j = (s_j, \psi_j) = \mathcal{T}^j z$ for $0 \le j < q$. Prove that z is q-periodic if and only if $d\mathcal{L}_q = 0$, i.e. $\partial_j \mathcal{L}_q(s_0, \cdots, s_{q-1}) = 0$ for every $0 \le j < q$. (Hint: Use Exercise 2.8).

We can now give a variational characterization of the length spectrum:

$$\mathrm{LS}(\Omega) = \bigcup_q \text{Critical Values of } \mathcal{L}_q.$$

The above allows to easily prove the following

Theorem 2.15. *The Length spectrum of a convex billiard has Lebesgue measure zero.*

Proof. Sard's Lemma states that the set of critical values of a C^n real-valued function on a d-dimensional manifold has measure zero provided that $n \ge d$. This immediately implies the result if $r = \infty$, by applying Sard's Lemma to the C^∞-functions $\mathcal{L}_q$ for arbitrary q. If $r < \infty$, we can argue as follows:

For any q, let $\Theta_q : \mathbb{T}^2_\ell \to \mathbb{T}^q_\ell$ be the map that sends (s_0, s_1) to the q-tuple $(s_0, s_1, \cdots, s_{q-1})$ obtained by collecting the next $q-2$ collision points of the billiard trajectory passing through the initial pair (s_0, s_1). Notice that the map Θ_q is as smooth as the billiard map, hence it is C^{r-1}. We now define the auxiliary map

$$\mathcal{L}^*_q(s_0, s_1) = \mathcal{L}_q(\Theta_q(s_0, s_1)).$$

Then, the set of critical values of $\mathcal{L}_q$ is the same as the set of critical values of $\mathcal{L}^*_q$; the latter is a real-valued C^{r-1} function of a 2-dimensional manifold; hence the result follows provided that $r \ge 3$. □

2.3. **Existence of periodic orbits.**

Here we show existence of periodic orbits of arbitrary (rational) rotation number. We begin by collecting some preliminary information about the set $\hat{\mathbb{T}}^q_\ell$. Given $(s_0, s_1, \cdots s_{q-1}) \in \hat{\mathbb{T}}^q_\ell$, we denote with $\alpha_i \subset \mathbb{T}_\ell$ the positively oriented open arc joining s_i with $s_{i+1 \mod q}$.

Lemma 2.16. *Fix $q \ge 2$; for any $\mathbf{s} = (s_0, s_1, \cdots, s_{q-1}) \in \hat{\mathbb{T}}^q_\ell$ there exists a (unique) $p \in \{1, \cdots, q-1\}$ so that $\forall s \in \mathbb{T}_\ell \setminus \{s_0, s_1, \cdots, s_{q-1}\}$ belongs to exactly p arcs α_i. Moreover, the map $\mathbf{s} \mapsto p$ is constant on each connected component of $\hat{\mathbb{T}}^q_\ell$.*

Proof. For simplicity, let us assume that $\ell = 1$ throughout this proof and denote $\mathbb{T} = \mathbb{T}_1$. Let $\Psi_{\mathbf{s}} : \mathbb{T} \to \mathbb{T}$ be a continuous function that maps each circle arc $[j/q, (j+1)/q]$ (homeomorphically and preserving orientation) to the closed arc $\bar{\alpha}_j \subset \mathbb{T}$. The map $\Psi_{\mathbf{s}}$ is a continuous circle map: let p denote its topological degree. We claim that $p \in \{1, \cdots, q-1\}$: in fact let $\tilde{\Psi}_{\mathbf{s}} : \mathbb{R} \to \mathbb{R}$ be a lift of $\Psi_{\mathbf{s}}$; then

$$p = \deg \Psi_{\mathbf{s}} = \tilde{\Psi}_{\mathbf{s}}(1) - \tilde{\Psi}_{\mathbf{s}}(0) = \sum_{j=0}^{q-1} \tilde{\Psi}_{\mathbf{s}}((j+1)/q) - \tilde{\Psi}_{\mathbf{s}}(j/q) = \sum_{j=0}^{q-1} |\alpha_j|,$$

where $|\alpha_j|$ denotes the length of the arc α_j. Since, by construction $0 < |\alpha_j| < 1$, we conclude that $0 < p < q$, which proves the claim. The above formula also implies that the map $\mathbf{x} \mapsto p$ is continuous in $\hat{\mathbb{T}}^q$, since for any j, the map $\mathbf{x} \mapsto |\alpha_j|$ is continuous in $\hat{\mathbb{T}}^q$. Since p assumes only integer values and is continuous, we gather that p is constant on each connected component of $\hat{\mathbb{T}}^q$. □

Definition 2.17. Let $q \in \mathbb{Z}_{\geq 2}$ and $p \in \{1, \cdots, q-1\}$; then define

$$\hat{\mathbb{T}}^{p,q}_\ell = \{\mathbf{s} = (s_0, s_1, \cdots, s_{q-1}) \in \hat{\mathbb{T}}^q_\ell \text{ s.t. } p(\mathbf{s}) = p\}.$$

Observe that if $(s_0, \cdots, s_{q-1}) \in \hat{\mathbb{T}}^{p,q}_\ell$ identifies a q-periodic point, then it is necessarily a periodic point of rotation number p/q: in fact, p is, by definition, the winding number of the orbit. For any $s \in \mathbb{T}$, then, we let

Lemma 2.18. *For any $\omega \in \mathbb{Q} \cap (0, 1/2]$ there exists at least one periodic point of rotation number ω.*

Proof. The lemma follows from a simple variational argument: let $\omega = p/q$ in lowest order. Notice that $\mathcal{L}_q$ is (uniformly) continuous on the closure $\overline{\hat{\mathbb{T}}^{p,q}_\ell}$ and –as noticed earlier– smooth on $\hat{\mathbb{T}}^{p,q}_\ell$. In particular, the restriction $\mathcal{L}_q|_{\overline{\hat{\mathbb{T}}^{p,q}_\ell}}$ has a maximum, attained for $\bar{\mathbf{s}} = (\bar{s}_0, \cdots, \bar{s}_{q-1})$. We claim that $\bar{\mathbf{s}} \notin \partial\hat{\mathbb{T}}^{p,q}_{\ell,s}$: in fact, assume otherwise by contradiction. By definition, $\partial\hat{\mathbb{T}}^{p,q}_\ell \subset \partial\hat{\mathbb{T}}^q_\ell$; hence for some j we would have $\bar{s}_j = \bar{s}_{j+1}$, but this contradicts the fact that $\bar{\mathbf{s}}$ is a maximum, since separating $\bar{s}_j$ from $\bar{s}_{j+1}$ would increase the length of the inscribed polygon (by the strict convexity of $\partial\Omega$). We conclude that $\bar{\mathbf{s}}$ is an interior maximum and thus

$$d\mathcal{L}_q(\bar{\mathbf{s}}) = 0$$

which, by definition, implies that $\bar{\mathbf{s}}$ is a periodic orbit. By earlier considerations we gather that its rotation number is p/q. □

Remark 2.19. The periodic orbits obtained by means of the above argument are called *maximal* or *Aubry-Mather* or *action minimizer*.

Remark 2.20. A celebrated refinement of the above result due to Birkhoff [6, 7] shows that indeed there exist at least 2 distinct periodic orbits for each rotation number; the other orbits are sometimes called *minimax* orbits. Birkhoff's result is sharp (e.g. a billiard on an ellipse has exactly 2 periodic orbits of rotation number $1/2$).

Remark 2.21 (Factoids about periodic orbits). It is entirely possible that a domain has more than 2 periodic orbit of given rotation number. In fact there can be arbitrarily many (e.g. finitely many, infinitely many, 1-parameter families.[11]) Observe that 1-parameter families of periodic orbits have necessarily the same length, but there are also examples of domains with a Cantor set of periodic orbits of the same rotation number, whose lengths also form a Cantor set[12].

[11] Remarkably, the presence of such "degenerate" families of periodic orbits can also be detected in the Wave Trace Formula.

[12] I am preparing a paper about this fact: such domains are in fact C^∞-dense

Here is an interesting collection of facts about a particular class of 1-parameter families called *caustics*.

- A disk is such that any inscribed regular (possibly non-convex) polygon draws the trajectory of a periodic orbit. For the disk we thus have "complete" 1-parameter families of periodic orbits of any rational rotation number. *Complete* means that the family is parametrized by $\mathbb{T}^1$; such families are also called *resonant caustics*. In fact this is an *if and only if* (see [4]).
- An ellipse has resonant caustics of any rotation number **except** $\omega = 1/2$.
- All curves of constant width have a resonant caustic for rotation number $1/2$.
- There is a C^∞-dense set of domains with the property that there exists a resonant caustic of rotation number $1/q$ for some q (see [24]).
- It is not known if there is a non-elliptic domain which admits more than 1 resonant caustics (except, of course) .
- Every (sufficiently smooth[13]) domain close to a disk which admits resonant caustics for every rotation number $1/q$ where $q > 2$ is an ellipse (see [2]).
- Every centrally symmetric C^2 domain which admits resonant caustics for every rotation number in $\mathbb{Q} \cap (0, 1/4]$ is an ellipse (see [5]).

Problems concerning the abundance of resonant caustics go under the umbrella term of *Birkhoff–Poritsky* conjecture. This is arguably among the most important conjectures in (low-dimensional) dynamics. In this language it could be stated as:

Conjecture (Birkhoff–Poritsky [29])**.** *Assume that Ω is so that there exists an open set $U \subset (0, 1/2)$ such that for any $\omega \in \mathbb{Q} \cap U$ there exists a resonant caustic of rotation number ω. Then Ω is an ellipse.*

The conjecture is wide open.

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2.4. Spectral Rigidity questions.

In this section we list some natural questions that arise for Birkhoff billiards; the most challenging is:

Question. Assume that $\mathrm{LS}(\Omega) = \mathrm{LS}(\Omega')$; is it true that Ω is isometric to Ω'?

It is not completely clear whether or not the above should be true; we can paraphrase Sarkak's conjecture:

Conjecture (Sarnak [26])**.** *For any $\Omega \in \mathcal{M}^\infty$, the set of Ω' that are isospectral to Ω is discrete (modulo isometries).*

As mentioned in Remark 2.12, rotation numbers can serve as a marking for the Length Spectrum, but the non-uniqueness described in the previous section means that the corresponding map $\mathrm{MLS} : \mathbb{Q} \cap (0, 1/2] \to \mathbb{R}_{>0}$ may be multi-valued; we can then extract a single valued function by considering the *maximal Marked Length Spectrum*:

$$\mathrm{MMLS} : \omega \mapsto \max \mathrm{MLS}(\omega)$$

and ask

Question. Assume that $\mathrm{MMLS}_\Omega = \mathrm{MMLS}_{\Omega'}$; is it true that Ω is isometric to Ω'?

---

[13] OK, this is embarassing: we need C^{39}.

Of course we can even ask the same question for MLS itself

Question. Assume that $\mathrm{MLS}_{\Omega} = \mathrm{MLS}_{\Omega'}$; is it true that Ω is isometric to Ω'?

An even weaker conjecture assumes that the billiard maps T_{Ω} and $T_{\Omega'}$ associated to domains Ω and Ω' are C^0-conjugate (recall that we mentioned "relative markings" in Lecture 1).

Question. Assume that the length of a periodic orbit in Ω equals the length of the corresponding periodic orbit in Ω'; is it true that Ω is isometric to Ω'?

In fact, Guillemin conjectured that a C^0-conjugacy of the billiard maps is enough to imply that two domains are homothetic. The conjecture is still open. If conjecture were true, then the above question would of course have an affirmative answer.

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Stepping aside from determination questions, and embracing the more accessible spectral rigidity questions, let us give a definition:

Definition 2.22. A domain Ω is said to be spectrally rigid (within a certain class) if any smooth isospectral family of domains Ω_t in the class is necessarily a trivial (i.e. isometric) family.

Question. Is it true that any smooth strongly convex domain is spectrally rigid?

An affirmative answer to the above question would be a step forward Sarnak's conjecture, but it would not imply it directly.

Exercise 2.23. Let $\mathbb{D}$ be the unit disk. Construct a sequence Ω_n of domains in $\mathcal{M}^\infty$ with the property that $\Omega_n \to \mathbb{D}$ in the natural topology, but there is no smooth deformation Ω_t of $\mathbb{D}$ so that there exists $t_n \to 0$ and $\Omega_n = \Omega_{t_n}$.

We will study (in Lecture 4) the proof of the following result:

Theorem 2.24 (—, Kaloshin, Wei [14]). *Consider the class of C^8 axially-symmetric domains; then a domain Ω that is sufficiently close to a disk is spectrally rigid (in the class of C^8 axially-symmetric domains).*

Lecture 3. Dispersing billiards and Lyapunov exponents

The class of hyperbolic billiards was considered a bit later in the game; the study of their ergodic properties was pioneered by Yakov Grigorevich Sinai and a particularly representative class of systems is called *Sinai billiards*. We here work with a related class of hyperbolic billiards which are called *open dispersing billiards*.

Definition 3.1. Let $r > 3$ (or $r = \infty$, or $r = \omega$), $N > 2$ and $(\mathcal{O}_i)_{i=0}^{N-1}$ be a finite, pairwise disjoint collection of C^r strongly convex domains, i.e. $\mathcal{O}_i \in \mathcal{M}^r$ which we call *scatterers*. We further assume the *non-eclipse condition*, that is: for any $i \neq j \neq k$, $i, j, k \in \{0, \cdots, N-1\}$ we require that $\mathrm{Hull}(\mathcal{O}_i \cup \mathcal{O}_j) \cap \mathcal{O}_k = \emptyset$. Then we call the unbounded region $\Omega = \mathbb{R}^2 \setminus \bigcup_{i=0}^{N-1} \mathcal{O}_i$ a *open dispersing billiard table*, or *hyperbolic billiard Table*. We denote the set of all C^r hyperbolic billiard tables with N scatterers by $\mathcal{B}_{\mathrm{H}}^{N,r}$

Let us denote with $\mathcal{N} = \{0, \cdots, N-1\}$; we define the phase space X_Ω as the disjoint union

$$X_\Omega = \bigsqcup_{i \in \mathcal{N}} \partial\mathcal{O}_i \times [-\pi/2, \pi/2];$$

Here the standard coordinates (x, φ) identify a point on the boundary of some scatterer and the angle φ, which we measure with respect to the normal at x pointing inside the domain (i.e. outside of the scatterer) and so that angle $-\pi/2$ corresponds to the positively oriented tangent.

We define $\iota : X_\Omega \to \mathcal{N}$ to be the map that sends each point (x, φ) to the index i of the scatterer $\mathcal{O}_i$ so that $x \in \partial\mathcal{O}_i$.

Given $z = (x, \varphi) \in X_\Omega$, let $\Lambda(x, \varphi)$ be the oriented line passing through x and making an angle φ with the positively oriented tangent vector at x; let i be the index of the scatterer $\mathcal{O}_i$ so that $x \in \partial\mathcal{O}_i$. By the non-eclipse condition, the line Λ will intersect at most one of the other scatterers. Let

$$X^+_\Omega = \{z \in X_\Omega \text{ s.t. } \Lambda(z) \text{ intersects another scatterer}\}.$$

For $(x, \varphi) \in X^+_\Omega$, we let x' be the point of intersection of Λ with the other scatterer that is closest to x. [14] We let φ' be the angle that Λ forms with the positively oriented vector tangent to $\partial\Omega$ at x'. Then we can define the *billiard map*:

Definition 3.2. The map $T_\Omega : X^+_\Omega \to X_\Omega$ given by $T_\Omega(x, \varphi) \mapsto (x', \varphi')$ is called the *Billiard map* associated to Ω.

We denote with $X^-_\Omega = T_\Omega X^+_\Omega \subset X_\Omega$. For $i, j \in \mathcal{N}$ and $i \neq j$, we define

$$L_{ij} : \partial\mathcal{O}_i \times \partial\mathcal{O}_j \to \mathbb{R}_{>0}$$

to be the Euclidean distance $L_{ij}(x, x') = \operatorname{dist}(x, x')$; let $\ell_i = |\partial\mathcal{O}_i|$ and assume we fixed arc-length parametrizations $\gamma_i : \mathbb{T}_{\ell_i} \to \partial\mathcal{O}_i$. **Notice** that since Ω is in the outside of each scatterer, we take parametrizations to be oriented clockwise.

Then, similarly as before, we define

$$\mathcal{L}_{ij}(s, s') = L(\gamma_i(s), \gamma_j(s')).$$

The collection of functions $\mathcal{L}_{\cdot\cdot}$ serve as generating functions[15] for the dynamics, in the sense that, taking $r = \sin\varphi$:

$$\mathcal{T}(s, r) = (s', r') \iff \begin{cases} r &= \partial_1 \mathcal{L}_{ij}(s, s') \\ r' &= -\partial_2 \mathcal{L}_{ij}(s, s'), \end{cases}$$

where $i = \iota(s)$ (resp. $j = \iota(s')$). For $i \neq j$ with $i, j \in \mathcal{N}$, we say that $(s, s') \in \mathbb{T}_{\ell_i} \times \mathbb{T}_{\ell_j}$ is *admissible* if the segment $\overline{\gamma_i(s_i)\gamma_j(s_j)}$ is disjoint from the interior of $\mathcal{O}_i$ and $\mathcal{O}_j$. Let $A_{ij} \subset \mathbb{T}_{\ell_i} \times \mathbb{T}_{\ell_j}$ be the set of admissible pairs.

Exercise 3.3. Prove the following facts:

[14] Observe that, by convexity of the scatterers, there are at most 2 points of intersection of any line with the boundary of any scatterer.

[15] Notice that there is a difference of sign with respect to Birkhoff billiards; this is due to the fact that we reverse orientations

- let $(s, s') \in \partial A_{ij}$; then the segment $\overline{\gamma_i(s)\gamma_j(s')}$ is tangent to at least one of the scatterers $\mathcal{O}_i$ or $\mathcal{O}_j$.
- if $(s, s') \in \mathrm{int} A_{ij}$, then the Hessian of $\mathcal{L}_{ij}$ is positive definite at (s, s').

3.1. Marking.

Let us define the sets

$$\Sigma_q = \{\sigma \in \mathcal{N}^q \text{ s.t. } \sigma_k \neq \sigma_{k+1 \mod q}\}$$
$$\Sigma = \{\sigma \in \mathcal{N}^{\mathbb{Z}} \text{ s.t. } \sigma_k \neq \sigma_{k+1}\}$$

For $\sigma \in \Sigma_q$, let

$$A_\sigma = \{\mathbf{s} \in \mathbb{T}_{\ell_{\sigma_0}} \times \mathbb{T}_{\ell_{\sigma_1}} \times \cdots \times \mathbb{T}_{\ell_{\sigma_{q-1}}} \text{ s.t. } (s_j, s_{j+1}) \in A_{\sigma_j\sigma_{j+1}} \text{ for any } j = 0, \cdots, q-1\}$$

Proposition 3.4. *Let $\sigma \in \Sigma_q$; there exists a unique q-periodic point $z = z(\sigma) \in X_\Omega$ so that $\iota(T^n z) = \sigma_{n \mod q}$.*

Proof. For $\sigma \in \Sigma_q$, let us define the function $\mathcal{L}_\sigma : \mathbb{T}_{\ell_{\sigma_0}} \times \mathbb{T}_{\ell_{\sigma_1}} \times \cdots \times \mathbb{T}_{\ell_{\sigma_{q-1}}} \to \mathbb{R}_{>0}$ given by:

$$\mathcal{L}_\sigma : (s_0, s_1, \cdots, s_{q-1}) = \sum_{j=0}^{q-1} \mathcal{L}_{\sigma_j\sigma_{j+1 \mod q}}(s_j, s_{j+1 \mod q}).$$

The restriction $\mathcal{L}_\sigma|_{A_\sigma}$ is a smooth function on a compact set; by Exercise 3.3, the function is strictly convex, and as such it admits a unique global minimum $\bar{\mathbf{s}} = (\bar{s}_0, \cdots, \bar{s}_{q-1}) \in A_\sigma$. If we show that $\bar{\mathbf{s}} \notin \partial A_\sigma$, we conclude that $\bar{\mathbf{s}}$ identifies points on $\partial\Omega$ with the property that the polygon with these points as vertices is so that each angle at such vertices is bisected by the normal to $\partial\Omega$ at the collision point. The proof then follows from an argument by contradiction... □

Exercise 3.5. Complete the missing details in the above proof.

Remark 3.6. The uniqueness stated above is analogous to the statement in Geometry about uniqueness of the geodesic corresponding to a given free homotopy class. The number of scatterers N corresponds to a "topological type" of billiard. The *shift map* $\sigma = (\sigma_k)_{k\in\mathbb{Z}} \mapsto \sigma' = (\sigma_{k+1})_{k\in\mathbb{Z}}$ serves as a *topological model* for the billiard map.

We can define a natural notion of distance on Σ by means of:

$$d(\sigma, \sigma') = \exp(-\max\{k \in \mathbb{Z} \text{ s.t. } \sigma_l = \sigma'_l \text{ for all } |l| \leq k\})$$

(and $d(\sigma, \sigma') = 0$ if $\sigma = \sigma'$). With the topology induced by this distance Σ is a Cantor set. There is a natural map $\Sigma_q \to \Sigma$ given by

$$(\sigma_j)_{j=0}^{q-1} \mapsto (\sigma_{j \mod q})_{j\in\mathbb{Z}}.$$

Now let $\Sigma_{\text{per}} = \bigcup_{q>1} \Sigma_q$, which we regard as a subset of Σ;

Lemma 3.7. *The map $\sigma \mapsto z(\sigma)$ defined by Proposition 3.4 on Σ_{per} is uniformly continuous, and can be extended to a map on Σ.*

The image on X of the map above is a (homeomorphic image of a) Cantor set identifying all points in phase space that correspond to orbits that never escape to ∞.

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We can now define the *Marked Length Spectrum* as follows. By Proposition 3.4, there is a 1-1 correspondence between periodic orbits for the billiard Ω and elements of Σ_{per}. We thus define

$$\mathrm{MLS}_\Omega : \Sigma_{\text{per}} \to \mathbb{R}_{>0}$$

to be the map that sends a sequence σ to the length $\mathcal{L}_\sigma(\bar{s}_0, \cdots, \bar{s}_{q-1})$ of the unique billiard orbit corresponding to σ. The length spectrum of Ω is just the image $\mathrm{MLS}_\Omega(\Sigma_{\text{per}}) \subset \mathbb{R}_{>0}$; it can be identified once again as a subset of the set of critical values of some functions. However, since Σ_{per} is countable, we conclude that for an hyperbolic billiard, $\mathrm{LS}(\Omega)$ is also countable. It is actually very easy to show that it is discrete:

Lemma 3.8. *Let $\Omega \in \mathcal{B}_H^{N,r}$, then for any $L > 0$ the set $LS(\Omega) \cap [0, L]$ is finite. In particular $LS(\Omega)$ is discrete.*

Proof. Let

$$l = \min_{i,j \in \mathcal{N}} \min_{x \in \partial\mathcal{O}_i, x' \in \partial\mathcal{O}_j} L_{ij}(x, x') = \min \mathrm{LS}(\Omega);$$

then any periodic orbit of prime period q has length at least ql; we conclude that $\mathrm{MLS}_\Omega \Sigma_q \subset [ql, \infty)$; hence

$$\#(\mathrm{LS}(\Omega) \cap [0, L]) \leq \sum_{q=2}^{L/l} \#\Sigma_q < \infty. \qquad \square$$

Exercise 3.9. Given $\sigma = (\sigma_0, \cdots, \sigma_{q-1}) \in \Sigma_q$, we define $\bar{\sigma} = (\sigma_0, \sigma_{q-1}, \sigma_{q-2} \cdots \sigma_1)$. Show that the orbit corresponding to σ and $\bar{\sigma}$ are the images of each other by the involution $\mathcal{I}$.

3.2. Questions about spectral determination.

A *naïve* spectral determination question would be as follows:

Question. Let $\Omega, \Omega' \in \mathcal{B}_{\mathrm{H}}^{N,r}$ so that $\mathrm{MLS}_\Omega = \mathrm{MLS}_{\Omega'}$; is it true that Ω is isometric to Ω'?

The answer to this question is negative, for a very good reason: let $H = \mathrm{Hull}(\bigcup_{j \in \mathcal{N}} \mathcal{O}_j)$; note that by the strict convexity of the scatterers and the non-eclipse condition, we have necessarily that $\partial H \cap \bigcup_{j \in \mathcal{N}} \partial\mathcal{O}_j$ is not empty, and moreover it contains a non-trivial arc of $\partial\mathcal{O}_j$ for any $j \in \mathcal{N}$. Let us call this arc *the dark side of the scatterer* $\mathcal{O}_j$ and denote it with α_j^Δ. Then let $x \in \partial H \cap \bigcup_{j \in \mathcal{N}} \partial\mathcal{O}_j$: by construction, any half-line issued from x with any angle φ will not intersect H (and thus any of the $\mathcal{O}_j$'s); in particular, the billiard map is not defined at (x, φ) for any φ and no periodic orbit can have x as a collision point. We conclude that any scatterer can be arbitrarily perturbed on their *dark side* without modifying The (Marked) Length Spectrum of Ω.

Remark 3.10. For any $i, j \in \mathcal{N}$, with $i \neq j$, let R_{ij} be the shortest line segment connecting $\partial\mathcal{O}_i$ with $\partial\mathcal{O}_j$; let x_{ij} be $R_{ij} \cap \partial\mathcal{O}_i$ be the foot of R_{ij} on $\partial\mathcal{O}_i$. Then $(x_{ij})_{j \neq i}$ partitions $\partial\mathcal{O}_i$ in $N - 1$ (open) arcs; only one of them contains the "dark side" α_i^Δ of $\mathcal{O}_i$; let us denote this arc by $\hat{\alpha}_i^\Delta$. It can be shown that any orbit that collides with any point on $\hat{\alpha}_i^\Delta$ will eventually (in the past or in the future) fly off to ∞. We conclude that no periodic orbit visits $\hat{\alpha}_i^\Delta$ and thus that any scatterer can be arbitrarily perturbed on $\hat{\alpha}_i^\Delta$ without modifying the (Marked) Length Spectrum of Ω.

Let us define the set

$$\beta_i = \overline{\{x \in \partial\mathcal{O}_i \text{ s.t. } \exists \text{periodic orbit colliding at } x\}}$$

By the above remark, we gather that $\beta_i \cap \hat{\alpha}_i^\Delta = \emptyset$; in principle β_i can be quite thin (e.g. a Cantor set), but it might also contain intervals. Let $\beta = \bigcup_i \beta_i$; then the strongest spectral determination that one can hope to prove in smooth regularity is:

Question. Let $\Omega, \Omega' \in \mathcal{B}_{\mathrm{H}}^{N,r}$ so that $\mathrm{MLS}_\Omega = \mathrm{MLS}_{\Omega'}$; is it true that Ω is isometric to Ω' *restricted on* β?

A simpler (but still open!) question concerns spectral rigidity in this set:

Question. Is it true that any domain $\Omega \in \mathcal{B}_{\mathrm{H}}^{N,r}$ is spectrally rigid on β? (i.e. any isospectral deformation of Ω is an isometry when restricted to β)?

In Lecture **??** we will see the proof of a related result, which, according to the tradition, assumes *analiticity* and *symmetry*, on top of *non-degeneracy*. These conditions greatly simplify the task at hand and allow to prove the following statement:

Theorem 3.11 (—, Kaloshin, Leguil [12])**.** *Consider the class of analytic domains $\mathcal{B}_H^{N,\omega}$ with the following additional properties:*

- *scatterer 1 and 2 are mirror image of each other and they are symmetric with respect to the 2-periodic orbit identified by the code* (12)
- *the Birkhoff Normal Form of the 2-periodic orbit* (12) *has nonzero quadratic coefficient (non-degenerate twist condition)*

Then if $MLS_\Omega = MLS_{\Omega'}$ for any two domains in this class, we conclude that Ω and Ω' are isometric.

Remark 3.12. The knowledge of the full Marked Length Spectrum is actually not necessary to recover the domain. It is sufficient to know the MLS in a "neighborhood" of (12). The non-degeneracy assumption seems to be merely technical and in principle it could be dropped (unless the Birkhoff sum is degenerate at all orders, but it is unknown if this is even possible for a billiard)

On the other hand, the symmetry assumption is crucial, at the moment.

The proof of the above theorem hinges on another result, which will be described in the following section 3.3; we will describe the result in full detail there, but a small teaser can be phrased as follows:

The marked length spectrum determines Lyapunov exponent of any given periodic orbit. This is an interesting statement, which is –to some extent– also known for convex billiards.

3.3. Recovering the Lyapunov spectrum.

Let x be a periodic point of prime period q for the billiard map; by Lemma 2.10, the differential $D_xT_\Omega^q \in \mathrm{SL}(2,\mathbb{R})$; we denote the (possibly complex) eigenvalues of $D_xT_\Omega^q$ with λ and λ^{-1}. We call

x a *hyperbolic periodic point* if $|\lambda| \neq 1$; in this case, conventionally, we choose λ so that $|\lambda|^{-1} < 1 < |\lambda|$.

Exercise 3.13. Show that any accumulation point of a sequence of periodic points of any given period is a periodic point (of the same period) so that $\lambda = \lambda^{-1} = 1$.

The above exercise in particular implies that hyperbolic periodic points of given period are discrete; more precisely, if x is an hyperbolic periodic point of period q, there exists a neighborhood $U \ni x$ so that every point $x' \in U \setminus \{x\}$ is not periodic of period q. (Periodic points of other periods will, in fact, accumulate on x). The dynamics in a neighborhood of an hyperbolic periodic point can be effectively described by its linearization; more precisely

Theorem 3.14 (Linearization). *Let x be a q-periodic point; then for any $\varepsilon > 0$, there exists a $C^{1+1/2}$-diffeomorphism $\Phi : U \to V$, where U is a neighborhood of $0 \in \mathbb{R}^2$ and V a neighborhood of x. so that*

$$T^q \circ \Phi = \Phi \circ dT^q \qquad\qquad \|d\Phi - Id\|_{C^0} < \varepsilon.$$

Every periodic orbit of a dispersing billiard is hyperbolic and, also, Birkhoff billiards have lots of hyperbolic orbits. In fact, the following holds:

Theorem 3.15. *Any periodic orbit obtained as a* quadratic *minimizer of the generating function (e.g. Aubry–Mather orbits in Lemma 2.18) is hyperbolic.*

Given a hyperbolic periodic point of prime period q, we define its *Lyapunov exponent* to be $\log|\lambda|/q \geq 0$. We collect all such exponents in an object that we call the *Lyapunov spectrum* of a billiard:

Definition 3.16. Let Ω be a billiard, then the *Lyapunov Spectrum* $\mathrm{LS}(\Omega)$ is defined as the set of all Lyapunov exponents of all periodic points. We define the *Marked Lyapunov Spectrum* MLS in the same way as we defined the Marked Length Spectrum.

We want to conclude this lecture with a relatively complete sketch of the proof of the following result

Theorem 3.17. *For a hyperbolic billiard $\Omega \in \mathcal{B}_H^{N,\infty}$, the map MLS : $\Sigma_p \to \mathbb{R}$ is a MLS-invariant.*

3.4. Proof of invariance of the Lyapunov spectrum.

First, we will prove that it is possible to recover the Lyapunov exponents of a special class of periodic orbits, called *palindromic*.

Definition 3.18. An orbit $\sigma \in \Sigma_q$ is called *palindromic* if $\bar{\sigma} = \sigma$.

Notice that the constraint $\sigma_j \neq \sigma_{j+1}$ implies that $q = 2p$. In turn this also implies that the cyclically translated orbit. $(\sigma_p\sigma_{p+1}\cdots\sigma_{q-1}\sigma_0\sigma_1\cdots\sigma_{p-1})$ is also palindromic. Since $\sigma = \bar{\sigma}$, by Exercise 3.9 we gather that palindromic orbit are invariant by the involution, that is to say that $\varphi_0 = 0$ (and $\varphi_p = 0$), that is to say the orbit has an orthogonal collision at σ_0 and σ_p.

As a first step of the proof of Theorem 3.17, we will show that we can determine the Lyapunov exponents for palindromic orbits. The key observation for palindromic orbits is

Lemma 3.19. *Let $(\bar{s}_0, \cdots, \bar{s}_{2p-1})$ denote the collision points of a palindromic periodic orbit σ of least period $2p$; let*

$$\hat{\mathcal{L}}(s_0, \cdots, s_p) = \sum_{j=0}^{p-1} \mathcal{L}_{\sigma_j \sigma_{j+1}}(s_j, s_{j+1})$$

Then

$$\hat{\mathcal{L}}(s_0, \cdots, s_p) - \hat{\mathcal{L}}(\bar{s}_0, \cdots, \bar{s}_p) = Q_\sigma(s_0 - \bar{s}_0, \cdots, s_p - \bar{s}_p) + R_\sigma(s_0 - \bar{s}_0, \cdots, s_p - \bar{s}_p)$$

where Q_σ is a positive definite quadratic form and R_σ is a cubic remainder

The proof of the above lemma follows from the strict convexity of $\mathcal{L}$ and is omitted. Using the above property, we show the following fundamental estimate

Lemma 3.20. *Let $\sigma = (\sigma_0, \cdots, \sigma_{q-1})$ be a palindromic periodic orbit of period $q = 2p$ and λ be the multiplier of DT^q at any point of the orbit. Let*

$$\tau = (\sigma_0', \sigma_1, \cdots, \sigma_{q-1})$$

be admissible (notice we just change the first index) so that $\tau\sigma$ is admissible and palindromic; then there exist nonzero constants C_0 and C_1 (depending on both σ and τ) so that

$$\mathit{MLS}(\tau\sigma^n) - n\mathit{MLS}(\sigma) = C_0 + C_1\lambda^n + o(\lambda^n)$$

Using the above lemma, we recover C_0, λ (and C_1) as MLS-invariant by taking the following limits:

$$\begin{aligned}
C_0 &= \lim_{n\to\infty} \mathrm{MLS}(\tau\sigma^n) - n\mathrm{MLS}(\sigma) \\
\log\lambda &= \lim_{n\to\infty} \frac{1}{n}\log(\mathrm{MLS}(\tau\sigma^n) - n\mathrm{MLS}(\sigma) - C_0) \\
C_1 &= \lim_{n\to\infty} (\mathrm{MLS}(\tau\sigma^n) - n\mathrm{MLS}(\sigma) - C_0)\lambda^{-n}
\end{aligned}$$

Proof of Lemma. The first observation is that, in the natural topology on Σ:

$$\lim_{n\to\infty} (\tau\sigma^n) = (\cdots\sigma\sigma\cdots\sigma\tau\sigma\cdots\sigma\sigma\cdots) =: h^\infty$$

The orbit h^∞ is an *heteroclinic orbit* for the orbit identified by σ; let us define

- $(y_k)_{k\in\mathbb{Z}}$ be the coordinates of the periodic orbit σ (notice that $y_k = y_{k+q}$ since we assume that the orbit is q-periodic);
- $(z_k^n)_{k\in\mathbb{Z}}$ be the coordinates of the periodic orbit $\tau\sigma^n$ (notice that $z_k^n = z_{k+q(n+1)}^n$);
- $(z_k^\infty)_{k\in\mathbb{Z}}$ be the coordinates of the heteroclinic orbit.

Using Theorem 3.14, we can show that, for sufficiently large n:

$$|z_{qk+j}^\infty - y_{qk+j}| \sim \lambda^{-k} \qquad \text{for all } k > 0, j = 0, \cdots, q-1 \tag{3.1}$$

$$|z_{qk+j}^n - z_{qk+j}^\infty| \sim \lambda^{n-k} \qquad \text{for all } 0 < k < n/2, j = 0, \cdots, q-1. \tag{3.2}$$

The crucial idea of the argument is the following: consider the homoclinic orbit z^∞, truncate it after qN collisions and compute the length of the truncated orbit:

$$L^\infty_{qN} = \sum_{k=0}^{N-1}\sum_{j=0}^{q-1}\mathcal{L}(z^\infty_{qk+j}, z^\infty_{qk+j+1}).$$

Since z^∞ is homoclinic to y, we can "approximate" each group of q collisions with the q-periodic orbit:

$$L^\infty_{qN} - N\mathrm{MLS}(\sigma) = \sum_{k=0}^{N-1}\sum_{j=0}^{q-1}(\mathcal{L}(z^\infty_{qk+j}, z^\infty_{qk+j+1}) - \mathcal{L}(y_j, y_{j+1 \mod q})).$$

s Now we split the inner sum in half (since σ is palindromic)

$$\begin{aligned}L^\infty_{qN} - N\mathrm{MLS}(\sigma) = \sum_{k=0}^{N-1}[&\hat{\mathcal{L}}(z^\infty_{qk},\cdots,z^\infty_{qk+p}) - \hat{\mathcal{L}}(y_0,\cdots,y_p)+\\ &\hat{\mathcal{L}}(z^\infty_{qk+p},\cdots,z^\infty_{qk+2p}) - \hat{\mathcal{L}}(y_p,\cdots,y_{2p}))]\end{aligned}$$

by Lemma 3.19, the right hand side is a geometric series (with ratio λ^2); we conclude that the left hand side has a limit as $N\to\infty$:

$$\tilde{C}_0 = \lim_{N\to\infty} L^\infty_{qN} - N\mathrm{MLS}(\sigma)$$

and that, asymptotically $L^\infty_{qN} - N\mathrm{MLS}(\sigma) - \tilde{C}_0$ is exponentially small with rate λ^2. Now, **if** L^∞_{qN} were MLS-invariant, we would be done; too bad they are not. The idea is then to approximate L^∞_{qN} with $\mathrm{MLS}(\tau\sigma^{2N-1})/2$

Observe, in fact, that, since $\tau\sigma^{2N-1}$ is palindromic

$$\begin{aligned}\frac{1}{2}\mathrm{MLS}(\tau\sigma^{2N-1}) &= \frac{1}{2}\sum_{k=0}^{2N-1}\sum_{j=0}^{q-1}\mathcal{L}(z^{2N-1}_{qk+j}, z^{2N-1}_{qk+j+1})\\ &= \sum_{k=0}^{N-1}\sum_{j=0}^{q-1}\mathcal{L}(z^{2N-1}_{qk+j}, z^{2N-1}_{qk+j+1})\end{aligned}$$

The quantity above should approximate L^∞_{qN}, hence we subtract $N\mathrm{MLS}\sigma$ from both sides:

$$\frac{1}{2}\mathrm{MLS}(\tau\sigma^{2N-1}) - N\mathrm{MLS}\sigma = \sum_{k=0}^{N-1}\sum_{j=0}^{q-1}(\mathcal{L}(z^{2N-1}_{qk+j}, z^{2N-1}_{qk+j+1}) - \mathcal{L}(y_j, y_{j+1 \mod q})).$$

Now the left hand side should converge to $\tilde{C}_0$, so we subtract also $\tilde{C}_0$ from both sides:

$$\begin{aligned}\frac{1}{2}\mathrm{MLS}(\tau\sigma^{2N-1}) - N\mathrm{MLS}\sigma - \tilde{C}_0 &= \sum_{k=0}^{N-1}\sum_{j=0}^{q-1}(\mathcal{L}(z^{2N-1}_{qk+j}, z^{2N-1}_{qk+j+1}) - \mathcal{L}(y_j, y_{j+1 \mod q}))-\\ &\quad - \sum_{k=0}^{\infty}\sum_{j=0}^{q-1}(\mathcal{L}(z^{\infty}_{qk+j}, z^{\infty}_{qk+j+1}) - \mathcal{L}(y_j, y_{j+1 \mod q}))\\ &= \sum_{k=0}^{N-1}\sum_{j=0}^{q-1}(\mathcal{L}(z^{2N-1}_{qk+j}, z^{2N-1}_{qk+j+1}) - \mathcal{L}(z^{\infty}_{qk+j}, z^{\infty}_{qk+j+1}))\\ &\quad - \sum_{k=N}^{\infty}\sum_{j=0}^{q-1}(\mathcal{L}(z^{\infty}_{qk+j}, z^{\infty}_{qk+j+1}) - \mathcal{L}(y_j, y_{j+1 \mod q})).\end{aligned}$$

Now, both quantities on the right hand side are geometric series by (3.1), and they sum up to something of order λ^N. This concludes the proof as we obtain the needed asymptotics for $n = 2N$. □

We omit the argument for non-palindromic orbits, but we can conclude in two ways

- either we adapt the argument (not too complicated, indeed)
- or we show that one can approximate the exponent of any orbit by the exponent of a palindromic orbit (for instance it is possible to show that, for arbitrary σ, the exponent of $(\sigma^n\tau'\bar{\sigma}^n\tau'')$ for suitable τ' and τ'' is asymptotic to the exponent of σ^{2n})

We now sketch the proof of our determination result

Sketch of the proof of Theorem 3.11. Theorem 3.17 convinced you that it is possible to recover information about the linearization of the Poincaré map (i.e. T^q) associated to any q-periodic point by using MLS-data. For analytic (smooth) maps, however, one can do quite a bit better: first of all it is possible to conjugate the Poincaré map in a neighborhood of a periodic point with its Birkhoff Normal Form by an analytic (smooth) change of coordinates. The Birkhoff Normal Form is given by:

$$(\xi, \eta) \mapsto (\lambda(\xi\eta)^{-1}\xi, \lambda(\xi\eta)\eta)$$

where $\lambda(\zeta)$ is an analytic function:

$$\lambda(\zeta) = \lambda_0 + \sum_{n=1}^{\infty} b_n\zeta^n$$

λ_0 is the largest multiplier of the periodic point (Lyapunov exponent) and b_n's are called *Birkhoff coefficients* and they are invariant by analytic (smooth) change of coordinates.

It is not hard to imagine that with more hard work one should be able to find asymptotics of MLS that allow to recover all Birkhoff coefficients of every periodic orbit. This is still not proved, but what we prove is:

Theorem 3.21 (—,Kaloshin, Leguil [12]). *Consider $\sigma = (12)$; if b_1 of the BNF associated to T^2 at σ is not zero, then b_n are MLS-invariants*

The condition about $b_1 \neq 0$ is the generic nondegeneracy condition that appears in the statement of Theorem 3.11; this is a *hyperbolic twist condition.*

Remark 3.22. We expect this result to hold for arbitrary σ and without the condition on b_1, but it seems to be quite a pain.

Once we have recovered the BNF, we resort to a variation of a well-known result by Colin de Verdière

Theorem 3.23 (see [9]). *The BNF of the Poincaré map of a periodic orbit with the symmetries we require* determines *the jet of the curvature function at the periodic points.*

The theorem is not hard to believe: the jet of the curvature function at the periodic points, by symmetry, allows to write the jet of the generating function of the dynamics near the orbit. The jet of the generating function in turns allows to write the BNF. In the proof, Colin de Verdiére observes that –given the symmetries– these constructions are "upper triangular" and so one can walk back and recover the jet from the BNF.

This, together with analiticity, allows to reconstruct scatterers 1 and 2 and with a bit more work (not too hard) one reconstructs all other scatterers. □

Lecture 4. Spectral rigidity for convex billiards

4.1. A functional analytic prelude.

We begin with a question that has nothing to do with dynamics. Let $f : \mathbb{T} \to \mathbb{R}$ be a smooth function. Assume that you know the following data:

$$\begin{aligned}
\ell_1(f) &= f(0) \\
\ell_2(f) &= \frac{1}{2}(f(0) + f(1/2)) \\
\cdots &= \cdots \\
\ell_q(f) &= \frac{1}{q}\sum_{j=0}^{q-1} f(j/q)
\end{aligned}$$

Question. What can you say about f? Can you reconstruct f?

If you look at the problem for long enough, it becomes clear that *odd functions* annihilate each of the ℓ's, so clearly you cannot reconstruct the odd part of f. Then you notice that $\ell_q \to \int f$, so you know the average of f. Can you reconstruct the even part of f?

Now, *this* is perhaps a bit trickier.

Lemma 4.1. *If f, g are even functions so that $\ell_j(f) = \ell_j(g)$ for any j, then $f = g$.*

Proof. Write f in Fourier series:

$$f(x) = \sum_{p\in\mathbb{Z}} \hat{f}_p e^{ipx}$$

then:

$$\ell_q(f) = \frac{1}{q}\sum_{j=0}^{q-1}\sum_{p\in\mathbb{Z}} \hat{f}_p e^{ipj/q} = \frac{1}{q}\sum_{p\in\mathbb{Z}} \hat{f}_p \sum_{j=0}^{q-1} e^{ipj/q} = \sum_{q|p} \hat{f}_p$$

The above is spelling out the fact that the sequence $\ell_q(f)$ is the *Möbius transform* of $\hat{f}_p$. It is a well-known result in analytic number theory that there is a way to invert the above relation that is called the *Möbius inversion formula*. Recall that a number $n \in \mathbb{Z}_{>0}$ is said to be *squarefree* if its prime decomposition has no repeated primes (e.g. $18 = 3\times 3\times 2$ is not squarefree, but $15 = 5\times 3$ is squarefree, and so is 30). Define the following function:

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is } \textbf{not} \text{ squarefree} \\ -1^{P(n)} & \text{otherwise, where } P(n) \text{ is the number of prime divisors of } n. \end{cases}$$

So, for instance $\mu(18) = 0, \mu(15) = 1$ and $\mu(30) = -1$. Then it is known that the following formula holds:

$$\hat{f}_p = \sum_{p|q} \mu(q/p)\ell_q(f).$$

The above means that we can *formally* recover the Fourier coefficients of f from the ℓ's. This formula can be used as long as one can make sense of the infinite sums, so we will need to require some decay in $\hat{f}$. The proof follows from the exercise below. □

Exercise 4.2. Assume f is so that it is zero average and $p^\gamma \hat{f}_p \to 0$ for some $\gamma > 1$, then $q^\gamma \ell_q(f) \to 0$ for the same γ.

Let us recast the above lemma in a form which will be useful later on: Let

$$h_\gamma = \{a_j \text{ s.t. } j^\gamma a_j \to 0\}$$

Then h_γ is a separable Banach space; let

$$X_\gamma = \{f : \mathbb{T} \to \mathbb{R} \text{ s.t. } f(x) = f(-x), \int_{\mathbb{T}} f = 0 \text{ and } \hat{f}_p \in h_\gamma\}.$$

Then the above discussion implies that:

Theorem 4.3. *If $\gamma > 1$, then the operator $T : X_\gamma \to h_\gamma$ which maps*

$$T : f \mapsto (\ell_q(f))_{q\in\mathbb{Z}_{>0}}$$

is invertible.

4.2. Smooth deformation of domains.

Definition 4.4. A family $(\Omega_\tau)_{\tau\in(-\varepsilon,\varepsilon)}$ of domains in $\mathcal{M}^r$ is said to be smooth if the following holds: let $\gamma(0,\cdot)$ be the arc-length parametrization of Ω_0; then we want the function $\gamma(\tau, s)$ to be smooth.

All results about rigidity hold modulo isometry; it makes thus sense to fix an origin and an orientation for all domains that we consider. We thus assume that $\gamma(\tau, 0) = 0$ and $\dot\gamma(\tau, 0) = (1, 0)$ for any τ. we also assume that the symmetry axis passes through the origin (and hence it is the vertical axis). We also assume that the parametrization in s is symmetric for any τ.

Given τ, we can define the *infinitesimal normal deformation* at s –denote it with $n_\tau(s)$– as the projection of $\partial_\tau\gamma$ on the *outer* normal vector at s. Assume that we can show that, if γ is an isospectral family, then for any τ $n_\tau(s)$ is identically zero; then we conclude that $\gamma(\tau, s)$ is constant in τ.

If γ is a family of smooth symmetric domains, then necessarily $n(s)$ is an even function of s. Notice moreover that since the point identified by $s = 0$ is constant in τ, we necessarily have $n(0) = 0$. Let P denote the space of smooth functions n that are even and so that $n(0) = 0$.

Given an orbit $\Theta = ((s_0, \psi_0), (s_1, \psi_1), \cdots, (s_{q-1}, \psi_{q-1}))$, it is easy to compute the variation of the length of the orbit by a normal perturbation $n(s)$.

Exercise 4.5. The (infinitesimal) variation of the length of the orbit Θ is given by:

$$\ell_\Theta(n) = \sum_{j=0}^{q-1} n(s_j) \sin\psi_j$$

Notice that the variation is a functional acting on the infinitesimal deformation.

Let us consider the following sequence of periodic orbits $(\Theta_q)_{q>1}$: Θ_q it the maximal[16] periodic orbit of rotation number $1/q$ which passes through the origin. It is easy to show that such orbits always exist, by symmetry (it is yet another variational argument)

The strategy is then as follows; let $\ell_q = \ell_{\Theta_q}$ be the functional associated to the orbit Θ_q. Assume that we can prove that if n is so that $\ell_q(n) = 0$, then n is identically 0. Then we have proven spectral rigidity. Given a symmetric domain Ω, we collect all functionals in an operator:$L_\Omega : P \to \mathbb{R}^\mathbb{N}$ which maps $n \mapsto (\ell_q(n))_{q>1}$. Assume we can find a suitable space of functions, which contains smooth functions, on which the operator L_Ω is injective. Then Ω is infinitesimally spectrally rigid, since only 0 is in the kernel. If we can show that it is injective for any Ω in a neighborhood, then we conclude that Ω is spectrally rigid. We call L_Ω the *linearized isospectral operator* associated to the domain Ω

4.3. The linearized isospectral operator.

Conjecture. *For* generic *symmetric $\Omega \in \mathcal{M}^\infty$, there exists an open set $U \ni \Omega$ so that for any $\Omega' \in U$ the operator $L_{\Omega'}$ is injective on smooth even functions.*

It is completely possible that the operator is injective for every (symmetric) domain, but this seems much harder to prove than the above conjecture. We do not have a single example of a domain for which L_Ω is not injective.

Remark 4.6. Injectivity of L_Ω is sufficient for rigidity, but not necessary. Assume that $0 \neq n \in \ker L_\Omega$. Possibly, there exists an orbit Θ' (not part of the list Θ_q) so that $\ell_{\Theta'}(n) \neq 0$.

The main result of [14] is indeed the following:

[16] It might be the case that there are several such orbits; in this case we select the one with smallest parameter s_1.

Theorem 4.7. *There exists a suitable space of perturbations P_*, which contains smooth even perturbations, and an open set $U \subset \mathcal{M}^8$ that contains all disks so that for any $\Omega \in U$, the operator L_Ω is injective on P_*.*

Remark 4.8. Indeed we show a bit more: we prove that the kernel of L_Ω is finite-dimensional for every Ω (and in a suitable neighborhood of a disk, the dimension of the kernel happens to be 0); this implies that if there is a symmetric domain that is not spectrally rigid, then the family of isospectral symmetric domains passing through it is at most finite-dimensional.

We now need to study properties of this operator. First, however, let us emphasize the relation with the problem that we encountered in Section 4.1.

Exercise 4.9. Show that if Ω is a disk of perimeter 1, then:

- $\Theta_q = ((j/q, \pi/q))_{j=0}^{q-1}$.
- ℓ_q is a multiple of the one defined in Section 4.1

Conclude that any disk is *infinitesimally spectrally rigid* with respect to smooth symmetric deformations (note that l_1 is missing, but the normalization takes care of it!)

The key thing to prove is that there are good coordinates which conjugate the dynamics to one which is close enough to the dynamics on the disk. To fix ideas (and our constants), we consider the disc of perimeter 1

The leading order is given by the so-called *Lazutkin parametrization*:

$$x(s) = C\int_0^s \rho^{-2/3}(s')ds',$$

where C is chosen so that $x(1) = 1$.

Remark 4.10. The Lazutkin parametrization was found in the 1970's by V. Lazutkin with the purpose of showing existence of KAM invariant curves for the Billiard problem. Lazutkin's result states that there exists a Cantor set of invariant curves (caustics corresponding to irrational rotation number) which accumulate on the boundary of the phase space.

The following is an interesting exercise:

Exercise 4.11. Let $n(x)$ be a normal perturbation expressed in the Lazutkin parametrization; then:

$$\int_{\mathbb{T}} n(x)dx = \partial_\tau |\partial\Omega_\tau|;$$

the average of n is the rate of change of the perimeter of the billiard table. Since the perimeter of the table is a length spectral invariant for deformations, we conclude that n (expressed in Lazutkin coordinates) must have 0-average.

Lazutkin coordinates are particularly nice to study the dynamics:

Lemma 4.12. *Let $r \geq 8$; for any $\varepsilon > 0$ there exists $\delta > 0$ so that for any Ω in a δ-C^r-neighborhood of the disc of perimeter 1, there exists smooth C^{r-4}-functions α*

(odd) and β (even) of C^{r-4}-norm smaller than ε so that:

$$x_q^j = j/q + \frac{\alpha(j/q)}{q^2} + \varepsilon O(q^{-4})$$
$$\frac{\sin\psi_q^j}{w_q(x_q^j)} = \frac{1}{q}\left[1 + \frac{\beta(j/q)}{q^2} + \varepsilon O(q^{-4})\right]$$

where w_q is a weight function[17]*. Moreover, α and β depend continuously on Ω (in the respective topologies).*

Remark 4.13. The functions α and β are related by a differential relation (roughly speaking, β is the derivative of α). They are both 0 if Ω is a disk.

Remark 4.14. If we choose sufficlently large r we can, in theory, find an arbitrarily precise expansion such as:

$$x_q^j = j/q + \sum_{k=1}^{n-1}\frac{\alpha_k(j/q)}{q^{2k}} + \varepsilon O(q^{-2n})$$
$$\frac{\sin\psi_q^j}{w_q(x_q^j)} = \frac{1}{q}\left[1 + \sum_{k=1}^{n-1}\frac{\beta_k(j/q)}{q^{2k}} + \varepsilon O(q^{-2n})\right]$$

where α_k and β_k are C^{r-2-2k}. We do not use the above expansion in our result, but it is good to know that we should have it for future work.

At this point we can indeed study the operator; inspired by our little exercise in analysis, we check what happens as $q \to \infty$; of course we obtain:

$$\lim_{q\to\infty} \ell_q(n) = \int_{\mathbb{T}} n(x)dx$$

Since we know that the right hand side should be zero for an isospectral deformation, we can restrict ourselves to zero Lazutkin average. We can then study the speed of decay and we find that

$$\lim_{q\to\infty} \frac{1}{q}\ell_q(n) = \partial_\tau I_2(\Omega_\tau)$$

where $I_k(\Omega)$ is the k-th Marvizi–Melrose invariant of the domain.

Remark 4.15. In their paper [25], Marvizi–Melrose show the existence of a sequence of Laplace-spectral invariants of any convex domains. Such invariants are obtained by integrating a differential polynomial in (some power of) the curvature function. In the same paper they show that such invariants can also be computed by the length spectrum and correspond to coefficients of the asymptotic expansion of lengths of periodic orbits of rotation number $1/q$.

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[17] If you *really* want to know it is

$$w_q(x) = q\sin\frac{\rho(x)^{-1/3}}{2Cq}$$

We then define a functional $\ell_\bullet(n) = \lim_{q\to\infty} \frac{1}{q}\ell_q(n)$ and further restrict to deformations which lie in the kernel of $\ell_\bullet$. Let

$$X^*_\gamma = \{n \in X_\gamma \text{ s.t. } \int n = 0, \ell_\bullet(n) = 0\},$$

Then, we can prove the following:

Lemma 4.16. *If $\gamma \in (3,4)$, then the operator L_Ω maps X^*_γ to h_γ; moreover it is continuous in Ω in the operator topology.*

Since L_Ω is invertible on X^*_γ if Ω is a disk, the above lemma allows to conclude that it is invertible in a neighborhood of a disk. Hence it is injective when restricted to smooth perturbations.

5. Conclusions

There is a very strong link between the quantum and the dynamical inverse problem. Both are very challenging, natural, interesting problems. Their study involves geometry, dynamics, analysis (and even a bit of number theory!). There are a number of open problems; let me recall them

Conjecture (Sarnak [26])**.** *Every domain bounded by a C^∞ curve is* locally *determined by its Laplace Spectrum.*

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