

THE MONOTONE CLASS THEOREM

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ABSTRACT. The proof of the Monotone Class Theorem is given in the book by Lieb–Loss but differs from the one I gave in class. Please find below the version that was given in class on Sept 13.

Recall the relevant definitions; here X is a non-empty set.

- an *algebra* of sets on X is a family of sets $\mathcal{A} \subset \mathcal{P}(X)$ that is closed under complement and finite unions
- an algebra is a σ -*algebra* if it is also closed under countable unions.
- a *monotone class* is a family of sets $\mathcal{C} \subset \mathcal{P}(X)$ with the property that the (countable) union of any increasing sequence of sets in \mathcal{C} is also in \mathcal{C} and the (countable) intersection of any decreasing sequence of sets in \mathcal{C} is also in \mathcal{C} .

Observe that any σ -algebra is a monotone class.

Lemma 0.1. *If a family \mathcal{F} is an algebra and a monotone class then it is indeed a σ -algebra.*

Proof. We need to show that countable union of any sequence of sets in \mathcal{F} belongs to \mathcal{F} . Let $\{E_k\}_{k \in \mathbb{N}} \subset \mathcal{F}$ be a sequence of sets; then construct the sequence

$$F_k = \bigcup_{i=1}^k E_k$$

Since \mathcal{F} is an algebra, $F_k \in \mathcal{F}$ for any $k \in \mathbb{N}$; moreover the sets F_k form an increasing sequence, therefore

$$\mathcal{F} \ni \bigcup_{k \in \mathbb{N}} F_k = \bigcup_{k \in \mathbb{N}} E_k.$$

We conclude that \mathcal{F} is a σ -algebra. □

Given any family $\mathcal{F} \subset \mathcal{P}(X)$ we can define the following objects

- the σ -*algebra* generated by \mathcal{F} , denoted by $\mathcal{M}(\mathcal{F})$ is the *smallest* σ -algebra of sets of X that contains \mathcal{F} ;
- the *monotone class* generated by \mathcal{F} , denoted by $\mathcal{C}(\mathcal{F})$ is the *smallest* monotone class that contains \mathcal{F} .

Notice that if $\mathcal{F} \subset \mathcal{M}(\mathcal{F}')$ (resp. $\mathcal{F} \subset \mathcal{C}(\mathcal{F}')$) we have $\mathcal{M}(\mathcal{F}) \subset \mathcal{M}(\mathcal{F}')$. Moreover, since any σ -algebra is a monotone class we immediately conclude that for any family \mathcal{F} , $\mathcal{C}(\mathcal{F}) \subset \mathcal{M}(\mathcal{F})$. The Monotone Class Theorem guarantees that if \mathcal{F} is an algebra, then this becomes an equality.

Theorem 0.2. *Let \mathcal{A} be an algebra of sets on X ; then $\mathcal{C}(\mathcal{A}) = \mathcal{M}(\mathcal{A})$.*

Proof. To ease notation, since \mathcal{A} is fixed let us write $\mathcal{C} = \mathcal{C}(\mathcal{A})$ and $\mathcal{M} = \mathcal{M}(\mathcal{A})$ without possibility of confusion.

First of all observe that it suffices to show that \mathcal{C} is a σ -algebra; then since $\mathcal{C} \supset \mathcal{A}$ and by the minimality of \mathcal{M} we can conclude that $\mathcal{M} \subset \mathcal{C}$; this, together with the trivial inclusion mentioned above concludes the proof. Observe moreover that by the above lemma it suffices to show that \mathcal{C} is an algebra; in order to do so we need to show that the class is invariant under finite unions and complements.

In order to show the closure by the complement operation, let us define

$$\mathcal{C}_* = \{E \in \mathcal{C} \mid X \setminus E \in \mathcal{C}\}$$

Observe that by design $\mathcal{C}_* \subset \mathcal{C}$; moreover since \mathcal{A} is an algebra we have that each element of \mathcal{A} will be in \mathcal{C}_* , therefore we have $\mathcal{A} \subset \mathcal{C}_*$. If we now show that \mathcal{C}_* is a monotone class, we can conclude –by the minimality of \mathcal{C} – that $\mathcal{C} \subset \mathcal{C}_*$, which would then imply that \mathcal{C}_* is the entire \mathcal{C} , i.e. \mathcal{C} is closed by the complement operation. The proof that \mathcal{C}_* is a monotone class is a routine check and follows from the definition [it is left to the reader]

Let us now show that the set \mathcal{C} is closed under finite unions. For any element $E \in \mathcal{C}$ we define

$$\mathcal{C}_E = \{F \in \mathcal{C} \mid E \cup F \in \mathcal{C}\} \subset \mathcal{C}$$

Observe that, by the symmetry of the definition we have $F \in \mathcal{C}_E$ if and only if $E \in \mathcal{C}_F$. Also, a routine check shows that \mathcal{C}_E is a monotone class for any choice of E [do it!]. If we manage to show that $\mathcal{A} \subset \mathcal{C}_E$ for any $E \in \mathcal{C}$, we will conclude that $\mathcal{C} \subset \mathcal{C}_E$, and thus $\mathcal{C} = \mathcal{C}_E$ for any E which shows closure of \mathcal{C} under finite unions.

In order to prove that $\mathcal{A} \subset \mathcal{C}_E$ for any $E \in \mathcal{C}$, let us now choose $A \in \mathcal{A}$; observe that for any $B \in \mathcal{A}$ we have $A \cup B \in \mathcal{A}$ (by definition of algebra), thus $A \cup B \in \mathcal{C}$, which implies that $B \in \mathcal{C}_A$. Since B was arbitrary we conclude that $\mathcal{A} \subset \mathcal{C}_A$ for any $A \in \mathcal{A}$. Since \mathcal{C}_A is a monotone class, by minimality of \mathcal{C} among all minimal classes containing \mathcal{A} , we conclude that $\mathcal{C}_A = \mathcal{C}$.

Otherwise said: for any $E \in \mathcal{C}$ and $A \in \mathcal{A}$ we have $E \in \mathcal{C}_A$. But by symmetry this implies that $A \in \mathcal{C}_E$ for any $E \in \mathcal{C}$, or $\mathcal{A} \subset \mathcal{C}$, that was what we wanted to show. \square