MAT1000HF Fall 2017
Midterm Practice Problems 2
Due Oct 30th

Problem 1
Let $(X, \mathcal{M}, \mu)$ be a measure space; show that for any $1 \leq p \leq q \leq r \leq \infty$, we have

$$L^q(\mu) \supset L^p(\mu) \cap L^r(\mu).$$

Problem 2
Let $C([0,1])$ be the vector space of continuous functions on $[0,1]$.

(a) Show that the function $\|\cdot\| : C([0,1]) \to \mathbb{R}$ defined by

$$\|f\| = \sup_{x \in [0,1]} |f(x)|$$

is a norm on $C([0,1])$.

(b) Show that the space $C([0,1])$ equipped with $\|\cdot\|$ is a Banach space.

(c) Let μ be a finite Borel measure on $[0,1]$ (i.e. $\mu([0,1]) < \infty$); show that

$$f \mapsto \Phi(f) = \int_0^1 f(x) d\mu(x)$$

is a *positive* continuous linear functional on $C([0,1])$ (recall that a positive functional is so that $\Phi(f) \geq 0$ if $f \geq 0$).

Problem 3
In the notation of the above problem, show that for any positive continuous linear functional $\Phi : C([0,1]) \to \mathbb{R}$ there exists a unique finite Borel measure μ so that $\Phi(f) = \int_0^1 f d\mu$. (Hint: suppose $u \in [0,1]$; define $F(u) = \lim_{\varepsilon \to 0} \Phi(\Theta_\varepsilon)$, where

$$\Theta_\varepsilon(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq u \\ 0 & \text{for } u + \varepsilon \leq x \end{cases}$$

and Θ_ε is linear between u and $u + \varepsilon$; show that F is increasing and right continuous; show that μ_F satisfies the required condition.)

Problem 4
Let X be a set and $x_0 \in X$ be a point. The function $\delta_{x_0} : \mathcal{P}(X) \to \mathbb{R}$ given by the following formula:

$$\delta_{x_0}(A) = \begin{cases} 1 & \text{if } x_0 \in A \\ 0 & \text{otherwise.} \end{cases}$$

Show that

$$\int_X f(x) d\delta_{x_0}(x) = f(x_0).$$