MAT1000HF Fall 2017

Midterm Practice Problems 3

Problem 1

Let $(X, \mathcal{M}, \mu)$ be σ-finite and $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions $f_n : X \to \mathbb{C}$; show that there exists a sequence of positive real numbers $(c_n)_{n\in\mathbb{N}}$ so that

$$\sum_{n=1}^{\infty} c_n f_n$$

converges almost everywhere.

First of all recall that the convergence of a series of real numbers $\sum_{n=0}^{\infty} a_n$ only depends on the behavior of the tail of the sequence; in other words, for any $\bar{n} \geq 0$ we have $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=\bar{n}}^{\infty} a_n$ converges.

Fix $\bar{a} > 0$ and let $a'_n = \min\{\bar{a}, a_n\}$; if $\sum a_n$ converges, then $a_n \to 0$ and so a'_n and a_n eventually agree, which implies that $\sum a'_n$ also converges. Likewise if $\sum a'_n$ converges, then $\sum a_n$ also converges.

Let us now prove the result: since X is σ-finite, we can find a disjoint sequence of sets $(E_n)_{n\in\mathbb{N}}$ of finite measure so that $X = \bigsqcup_n E_n$. Consider now the (nearly simple) function

$$\varphi(x) = \sum_n \frac{1}{n^2 \mu(E_n)} \chi_{E_n}$$

Notice that $\varphi \in L^1(\mu)$, since $\sum_n n^{-2}$ converges. Observe that for any $c > 0$ and $n \in \mathbb{N}$, the function $\min\{c|f_n|, \varphi\} \in L^1(\mu)$, as it is non-negative and bounded above by $\varphi \in L^1(\mu)$. Moreover, for any $x \in X$:

$$\lim_{c\to 0} \min\{c|f_n(x)|, \varphi(x)\} = 0$$

We conclude by the Dominated Convergence Theorem that for any $n \in \mathbb{N}$

$$\lim_{c\to 0} \int_X \min\{c|f_n(x)|, \varphi(x)\} = 0$$

This implies that for any $n \in \mathbb{N}$ we can choose c_n so that $\int_X \min\{c_n|f_n(x)|, \varphi(x)\} \leq n^{-2}$, which in turn imply that

$$\sum_n \int_X \min\{c_n|f_n(x)|, \varphi(x)\} \text{ converges}$$

By Folland, Theorem 2.25, this means that $\sum_n \min\{c_n|f_n(x)|, \varphi(x)\}$ converges a.e., but our previous discussion implies that this series converges at some point x if and only if $\sum_n c_n|f_n(x)|$ converges. This concludes the proof.