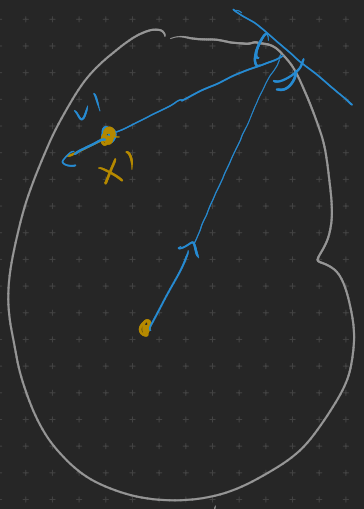
Laplace and Length rigidity

• Introduction

Domains in $\mathbb{R}^n$ — Closed Riemannian Mfolds (M, g)

Ω



1. Geodesic / Billiard flow on $X = T^1\Omega$ or $T^1_{(M,g)}$

let $x \in X$, $v \in T^1_x X$

$$\phi_t : (x, v) \longmapsto (x', v')$$

$$\Phi : \mathbb{R} \times T^1X \longrightarrow T^1X$$

t

$$\Phi_s \circ \Phi_t = \Phi_{t+s} \quad \forall\, t, s \in \mathbb{R}$$

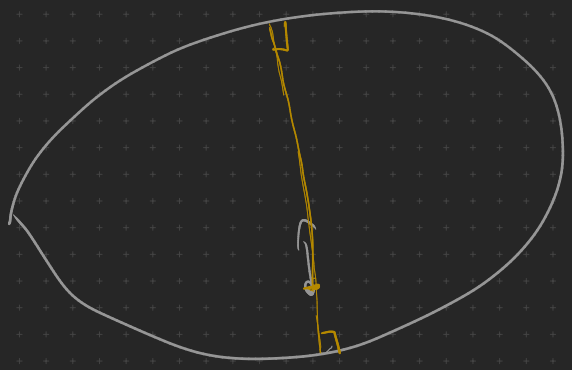
flow.

Stable orbits $\rightsquigarrow$ Periodic geodesics

Def (x,v) is T-periodic if

$$\phi_T(x,v) = (x,v)$$

example on S^2



$$LS(X) = \{ \text{lengths of closed geodesics on } X \} \subset \mathbb{R}_{>0}$$

Length spectrum of X (w/o info about multiplicity)

Q: How much info about X is coded in $LS(X)$?

Length rigidity

2. Laplace (– Beltrami) operator $-\Delta : C^2(X) \to C(X)$

$$-\Delta_{\mathbb{R}^n} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} = \text{div grad}$$

find $u \in C^\infty$

$$\begin{cases} -\Delta_x u = \lambda u \\ u = 0 \text{ on } \partial X \end{cases}$$ Dirichlet BC

$$Sp\Delta_X = \{0 \le \lambda_0 \le \lambda_1 \le \cdots \le \lambda_n \le \cdots\} \subset \mathbb{R}$$

Consider multiplicities

Q: How much info about X is contained in $Sp\Delta$? Laplace Rigidity

Note: Very strong relation b/w $Sp\Delta(X)$ and $LS(X)$

e.g. Wave trace formula:

$$\text{sing supp}\left(\sum_{\lambda \in Sp\Delta} e^{it\sqrt{\lambda}}\right) \subset \overline{\mathbb{Z} \cdot LS(X)} \cup \{0\}$$

$$\sum_{\lambda \in Sp\Delta} e^{it\sqrt{\lambda}} = \text{Tr}\left(e^{it\sqrt{\Delta}}\right)$$ Trace of Wave operator

Notions of Rigidity: Fix a class $\mathcal{Q}$ of Domains / Mfolds (e.g. planar domains / symetries / Mfolds of dim 4 w/ neg. curvature)

- $X \in \mathcal{Q}$ is **spectrally determined** if $\forall\, X' \in \mathcal{Q}$ with $Sp_\Delta(X) = Sp_\Delta(X')$ then $X \approx X'$ (≈ isometric)

- $X \in \mathcal{Q}$ is **locally sp. det.** if $\exists\, \mathcal{U} \ni X$ (with $\mathcal{U} \subset \mathcal{Q}$) s.t. $\forall\, X' \in \mathcal{U}$ w/ $Sp_\Delta(X) = Sp_\Delta(X')$ then $X \approx X'$

- $X \in \mathcal{Q}$ is **spec. rigid** if $\forall$ $t \longmapsto X_t$ is a 1-parameter family of objects.

  $X_0 = X$ if $\forall t$ $Sp_\Delta(X_t) = Sp_\Delta(X)$ then $X_t \approx X_0$

Note: Some definitions hold for $Sp_\Delta \hookrightarrow LS$

Some Prior Results.

- 1976 Guillemin, Kazhdan — Q = Neg. curved surfaces $d=2$; $\forall X \in Q$ is Δ-spec. rigid, LS ——

1998 Croke, Sharafutdinov — $d \geq 2$

- 1990 Otal $\perp$ Croke — MARKED Length spectrum determination of -'ly curved surfaces. $d=2$

2018 Guillarmou, Lefevre — MLS local det if $d>2$

- 2008 Zelditch — $\mathbb{Z}_2$-symetric, ANALYTIC $d=2$ are Δ-spec. determined

2016 —, Kolosin, Wei — $\mathbb{Z}_2$-symetric smooth domains close to Disk are LS-rigid